Boundary control of infinite dimensional irreversible port-Hamiltonian systems: the heat equation.

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1. INTRODUCTION

In this discussion paper we consider the (boundary) control of irreversible thermodynamic systems using the irreversible port Hamiltonian framework. We first show how infinite dimensional port-Hamiltonian formulations initially derived for reversible systems (Le Gorrec et al., 2005) have been extended to the modelling of irreversible thermodynamic systems controlled at the boundaries of their spatial domains (Ramírez et al., 2022). In a second instance we show, on the heat equation example, how to adapt the well known control by interconnection and damping injection method to the boundary control of this particular class of systems.

2. IRREVERSIBLE PORT HAMILTONIAN SYSTEMS

Thermodynamic systems are systems for which the thermal domain plays a central role and the energy goes from one physical domain to the thermal domain in an irreversible way. It is the case for example of chemical reactors, diffusion systems, smart materials, and all temperature dependent systems. This class of irreversible systems does not fit in the port Hamiltonian framework. In order to overcome this issue, and nevertheless exploit as far as possible the physical properties of the system, many alternative approaches such as contact formulations, pseudo port Hamiltonian formulations and GENERICs (Grmela and Öttinger, 1997) have been proposed in the litterature. Among them the irreversible port Hamiltonian (IPH) formulations (Ramírez et al., 2013) have shown to be very useful for analysis and control design (Ramírez et al., 2016) in the finite dimensional case. These formulations have been recently extended to the modelling of infinite dimensional irreversible thermodynamic systems controlled at the boundaries of their spatial domains, leading to the following definition (Ramírez et al., 2022).

Definition 1. A boundary controller irreversible port Hamiltonian system is a system defined by the following set of PDEs:

$$\frac{\partial}{\partial t}\begin{bmatrix} x(t,z)\\ s(t,z)\end{bmatrix} = \begin{bmatrix} P_0 & G_0\mathbf{R_0}(\mathbf{x}) \\ -\mathbf{R_0}(\mathbf{x})^\top G_0^\top & 0\end{bmatrix}\begin{bmatrix}\frac{\delta H}{\delta x}(t,z)\\ \frac{\delta H}{\delta s}(t,z)\end{bmatrix} + \begin{bmatrix} P_1\frac{\partial(.)}{\partial z} & \frac{\partial\left(G_1\mathbf{R_1}(\mathbf{x}).\right)}{\partial z} \\ \mathbf{R_1}(\mathbf{x})^\top G_1^\top \frac{\partial(.)}{\partial z} & g_s\mathbf{r_s}(\mathbf{x})\frac{\partial(.)}{\partial z} + \frac{\partial\left(g_s\mathbf{r_s}(\mathbf{x}).\right)}{\partial z}\end{bmatrix}\begin{bmatrix}\frac{\delta H}{\delta x}(t,z)\\ \frac{\delta H}{\delta s}(t,z)\end{bmatrix} \tag{1}$$

with $z \in [a, b]$, $x \in \mathbb{R}^n$ the set of energy variables, $s \in \mathbb{R}$ the entropy, $H(x, s)$ the total energy. $P_0 = -P_0^\top \in \mathbb{R}^{n\times n}$, $P_1 = P_1^\top \in \mathbb{R}^{n\times n}$, $g_s \in \mathbb{R}$, $G_0 \in \mathbb{R}^{n\times m}$, $G_1 \in \mathbb{R}^{n\times m}$ with m the number of states involved in the entropy production. $\mathbf{R_0} \in \mathbb{R}^{m\times 1}$, $\mathbf{R_1} \in \mathbb{R}^{m\times 1}$ and $r_s \in \mathbb{R}$ stand for the vectors of modulated driving forces with [1] $R_{k,i} = \gamma_{k,i}\left(x, z, \frac{\delta H}{\delta x}\right)\{S|G_k(:,i)|H\}, k \in \{0, 1\}$ and $r_s = \gamma_s\left(x, z, \frac{\delta H}{\delta x}\right)\{S|H\}$ with $\gamma_{k,i}\left(x, z, \frac{\delta H}{\delta x}\right), \gamma_s\left(x, z, \frac{\delta H}{\delta x}\right) : \mathbb{R}^n \to \mathbb{R}$, $\gamma_{k,i}, \gamma_s \geq 0$, non-linear positive functions. The input/output are given by

$$u(t) = W_B\begin{bmatrix} e_e(t,b)\\ e_e(t,a)\end{bmatrix}\ , \ \ y(t) = W_C\begin{bmatrix} e_e(t,b)\\ e_e(t,a)\end{bmatrix} \tag{3}$$

where

$$e_e(t,z) = \begin{bmatrix}\frac{\delta H}{\delta x}(t,z)\\ \mathbf{R}(\mathbf{x})\frac{\delta H}{\delta s}(t,z)\end{bmatrix} \text{ with } \mathbf{R}(\mathbf{x}) = \begin{bmatrix} 1\\ \mathbf{R_1}(\mathbf{x})\\ \mathbf{r_s}(\mathbf{x})\end{bmatrix} \tag{4}$$

and

$$W_B = \left[\frac{1}{\sqrt{2}}\left(\Xi_2 + \Xi_1 P_{ep}\right)M_p \ \ \frac{1}{\sqrt{2}}\left(\Xi_2 - \Xi_1 P_{ep}\right)M_p\right],$$
$$W_C = \left[\frac{1}{\sqrt{2}}\left(\Xi_1 + \Xi_2 P_{ep}\right)M_p \ \ \frac{1}{\sqrt{2}}\left(\Xi_1 - \Xi_2 P_{ep}\right)M_p\right],$$

where $M_p = \left(M^\top M\right)^{-1}M^\top$, $P_{ep} = M^\top P_e M$ and M is spanning the columns of P_e, defined by [2] $P_e =$

[1] The following pseudo (locally defined) brackets are used to define the thermodynamic driving forces of the system

$$\{\Gamma|\mathcal{G}|\Omega\} = \begin{bmatrix}\frac{\delta\Gamma}{\delta x} & \frac{\delta\Gamma}{\delta s}\end{bmatrix}\begin{bmatrix} 0 & \mathcal{G}\\ -\mathcal{G}^* & 0\end{bmatrix}\begin{bmatrix}\frac{\delta\Omega}{\delta x}\\ \frac{\delta\Omega}{\delta s}\end{bmatrix},$$
$$\{\Gamma|\Omega\} = \frac{\delta\Gamma}{\delta s}^\top\left(\frac{\partial}{\partial z}\frac{\delta\Omega}{\delta s}\right) \tag{2}$$

for some smooth functions Γ, Ω and $\mathcal{G}$.

[2] 0 has to be understood as the zero matrix of proper dimensions.

$\begin{bmatrix} P_1 & 0 & G_1 & 0 \\ 0 & 0 & 0 & g_s \\ G_1^\top & 0 & 0 & 0 \\ 0 & g_s & 0 & 0 \end{bmatrix}$ and where Ξ_1 and Ξ_2 satisfy $\Xi_2^\top \Xi_1 + \Xi_1^\top \Xi_2 = 0$ and $\Xi_2^\top \Xi_2 + \Xi_1^\top \Xi_1 = I$.

As an example the heat equation defined on a one dimensional spatial domaine ($z \in [0, L]$) can be formulated as an irreversible port Hamiltonian system choosing the entropy $s(z,t)$ as state variable and the total internal energy $U(t) = \int_0^1 u(z,t)dz$ where $u(z,t)$ is the internal energy density as Hamiltonian. From the balance equation on the internal energy and Gibb's equation one can write the IPH formulation

$$\frac{\partial s}{\partial t} = \frac{\partial}{\partial z}\left(\frac{\lambda}{T}\frac{\partial T}{\partial z}\right) + \frac{\lambda}{T^2}\left(\frac{\partial T}{\partial z}\right)^2 \tag{5}$$

where λ denotes the heat conduction coefficient. From (3) the boundary inputs and outputs of the system are

$$v(t) = \begin{bmatrix} \left(\frac{\lambda}{T}\frac{\partial T}{\partial z}\right)(t, L) \\ -\left(\frac{\lambda}{T}\frac{\partial T}{\partial z}\right)(t, 0) \end{bmatrix}, \quad y(t) = \begin{bmatrix} T(t,L) \\ T(t,0) \end{bmatrix}, \tag{6}$$

respectively the entropy flux and the temperature at each boundary.

3. BOUNDARY CONTROL OF THE HEAT EQUATION

We consider now the boundary control of the 1D heat equation. The idea is to use the Thermodynamic availability function $\mathcal{A} = \int_0^1 a(z,t)dz$, defining the distance between the energy and the tangent plane at the desired equilibrium point as shown in Figure 1 as closed loop Lyapunov function (Availability Based Interconnection (ABI)) and to use Entropy Assignment (EA) to guarantee the convergence of trajectories to the desired equilibrium.

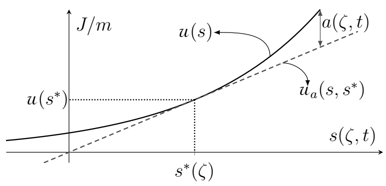

Fig. 1. Thermodynamic availability function.

In this respect the boundary control feedback $v = \beta(\mathbf{y}) + v'$, with v' an auxiliary boundary input, is chosen to map (6), (7) into the target system

$$\partial_t s = \overline{r}_s \partial_\zeta (\delta_s \mathcal{H}) + \partial_\zeta (\overline{r}_s \delta_s \mathcal{H}) \tag{7}$$

$$\tilde{\mathbf{u}} = \Xi v' \tag{8}$$

where $\mathcal{H} = U$ and

$$\Xi = \begin{bmatrix} \left.\frac{\delta_s \mathcal{A}}{T}\right|_L & 0 \\ 0 & \left.\frac{\delta_s \mathcal{A}}{T}\right|_0 \end{bmatrix} \text{ and } v' = \begin{bmatrix} \left.\lambda\left(\frac{\partial_\zeta(\delta_s \mathcal{A})}{T}\right)\right|_L \\ \left.\lambda\left(\frac{\partial_\zeta(\delta_s \mathcal{A})}{T}\right)\right|_0 \end{bmatrix} \tag{9}$$

and $\overline{r}_s = \gamma_s\{\mathcal{S}|\mathcal{A}\}$. It is the case if the following matching conditions are satisfied

$$\gamma_s\{\mathcal{S}|\mathcal{H}_a\}\partial_\zeta(\delta_s \mathcal{H}) + \partial_\zeta(\gamma_s\{\mathcal{S}|\mathcal{H}_a\}\delta_s \mathcal{H}) = 0 \tag{10}$$

$$\beta(\mathbf{y}) + \begin{bmatrix} \left.\lambda\left(\frac{\partial_\zeta(\delta_s \mathcal{H}_a)}{T}\right)\right|_L \\ \left.\lambda\left(\frac{\partial_\zeta(\delta_s \mathcal{H}_a)}{T}\right)\right|_0 \end{bmatrix} = 0 \tag{11}$$

A target temperature profile of the form $T_e^* = m^*\zeta + b^*, \forall \zeta \in [0, L]$ leads to the solution $\beta(\mathbf{y}) = \begin{bmatrix} \frac{km^*}{T|_L} & \frac{km^*}{T|_0} \end{bmatrix}^T$.

We consider now the additional feedback on (8)-(9)

$$\tilde{\mathbf{u}} = -\Gamma \mathbf{y} \tag{12}$$

with $\Gamma = \Xi \Phi \Xi^\top$, and $\Phi = \Phi^\top > 0$, then the system is asymptotically stable. If Φ is defined by $\Phi = diag\left(\frac{\phi_L}{T|_L}, \frac{\phi_0}{T|_0}\right)$ where ϕ_L and ϕ_0 are strictly positive, the target temperature profile is achievable from any initial condition T_0. At the end the control is

$$\mathbf{u} = \beta(\mathbf{y}) - \Phi \Xi^\top \mathbf{y} \tag{13}$$

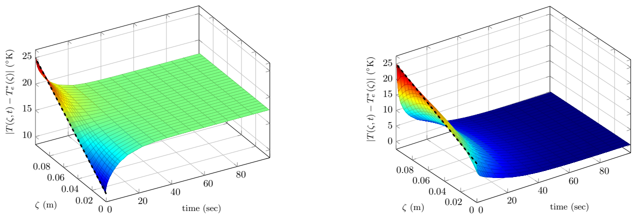

Fig. 2. Behavior of the absolute error of temperature response with respect to desired equilibrium profile, using ABI (left) control and ABI-EA (right) control.

As numerical application let's consider the heat equation with initial condition $T_0 = 303.15, \forall \zeta \in [0, 0.1]$ and target profile $T_e^* = 150\zeta + 313.15, \quad \zeta \in [0, 0.1]$. The closed loop performances using Availability based interconnection with or without Entropy assignment are given in Figure 2. It shows that the use of the availability based interconnection allows to reach an equilibrium but that EA is necessary to avoid bias.

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