A solution to shape optimization problems using time evolution equations

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1. INTRODUCTION

This paper deals with a numerical analysis method based on time evolution equations for solving nonparametric boundary shape optimization problems of domain boundaries. Shape optimization problems often appear at the final stage of design.

2. SHAPE OPTIMIZATION PROBLEM

Let $\Omega \subset \mathbb{R}^d,\ d = 2, 3$ be a design domain with boundary $\partial\Omega = \Gamma_{\mathrm{N}} \cup \Gamma_{\mathrm{D}}$ where Γ_{N} is a Neumann boundary and Γ_{D} is a Dirichlet boundary, $D = \cup\Omega$ be a set of design domains. We denote that $\boldsymbol{n} = (n_1, \cdots, n_d)$ is an outward normal unit vector on the boundary $\partial\Omega$, $\mathbf{0}$ is a zero vector and $\boldsymbol{\nabla} = (\partial/\partial x_1, \cdots, \partial/\partial x_d)$ for a point $\boldsymbol{x} = (x_1, \cdots, x_d)$ in Ω. We define the steady-state heat conduction problem:

$$\begin{aligned} &-\boldsymbol{\nabla}\cdot(q\boldsymbol{\nabla}u) = b, \quad \text{in} \quad \Omega, \\ &-(q\boldsymbol{\nabla}u)\cdot\boldsymbol{n} = p, \quad \text{on} \quad \Gamma_{\mathrm{N}}, \\ &u = u_{\mathrm{D}}, \quad \text{on} \quad \Gamma_{\mathrm{D}}, \end{aligned} \tag{1}$$

where $b \in \mathbb{R}$, $p \in \mathbb{R}$ and $u_{\mathrm{D}} \in \mathbb{R}$ are given functions, $q > 0$ is a given constant. In the problem (1), b presents a heat source in Ω, p presents a heat flux on Γ_{N}, u_{D} presents a temperature defined on the boundary Γ_{D} and q presents a thermal conductivity. Using the solution u to the problem (1), we define the objective function as a thermal resistance presented by

$$f_0(\Omega) = \int_\Omega bu\mathrm{d}\boldsymbol{x} + \int_{\Gamma_{\mathrm{N}}} pu\mathrm{d}\gamma - \int_{\Gamma_{\mathrm{D}}} u_{\mathrm{D}}\,(q\boldsymbol{\nabla}u)\cdot\boldsymbol{n}\mathrm{d}\gamma \tag{2}$$

and the constraint function as a volume for domain measure such as

$$f_1(\Omega) = \int_\Omega 1\mathrm{d}\boldsymbol{x}. \tag{3}$$

A shape optimization problem is defined by

$$\min_{\Omega}\left\{f_0(\Omega); f_1(\Omega) \le f_1(\Omega_0),\ u \text{ is a solution to (1)}\right\}, \tag{4}$$

where $\Omega_0 \in D$ is a given initial domain with boundary $\partial\Omega_0 = \Gamma_{\mathrm{M}} \cup \Gamma_{\mathrm{F}}$ and $f_1(\Omega_0)$ is the initial value of f_1 given as

$$f_1(\Omega_0) = \int_{\Omega_0} 1\mathrm{d}\boldsymbol{x}.$$

Here Γ_{M} is a moving boundary, i.e., Γ_{M} is deformed in the computational steps of optimization process and Γ_{F} is a fixed boundary, i.e., Γ_{F} is fixed in the computational steps of optimization process.

A Lagrangian function of the problem (4) is given as

$$\mathscr{L}(\Omega, u, v_0, v_1, \lambda_1) = \mathscr{L}_0(\Omega, u, v_0) + \lambda_1\mathscr{L}_1(\Omega, u, v_1), \tag{5}$$

where $\mathscr{L}_i(\Omega, u, v_i),\ i = 0, 1$ are Lagrangian functions for $f_i,\ i = 0, 1$, defined as

$$\mathscr{L}_i(\Omega, u, v_i) = f_i + \int_\Omega \{\boldsymbol{\nabla}\cdot(q\boldsymbol{\nabla}u) + b\}\, v_i\mathrm{d}\boldsymbol{x}. \tag{6}$$

respectively. Here $v_i,\ i = 0, 1$ are the Lagrange multipliers for the problem (1).

3. SHAPE DERIVATIVE

In order to solve the problem (4) by gradient based method, the shape gradient for the objective function f_0 and the constraint function f_1 with respect to the variation of Ω are requested. The shape gradient $\boldsymbol{g}_i,\ i = 0, 1$ can be obtained using the stationary conditions of $\mathscr{L}_i,\ i = 0, 1$. The shape gradient of $f_i,\ i = 0, 1$ are represented as

$$\frac{\mathrm{d}\mathscr{L}_i(\Omega, u, v_i)}{\mathrm{d}\Omega} = \frac{\partial\mathscr{L}_i}{\partial u}\frac{\partial u}{\partial\Omega} + \frac{\partial\mathscr{L}_i}{\partial v_i}\frac{\partial v_i}{\partial\Omega} + \frac{\partial\mathscr{L}_i}{\partial\Omega} \tag{7}$$

where $\partial u/\partial\Omega = \delta u$ is a variation of u, $\partial v_i/\partial\Omega = \delta v_i$ are variations of $v_i,\ i = 0, 1$.

The stationary condition of $\mathscr{L}_i,\ i = 0, 1$ for all variations δv_i of $v_i,\ i = 0, 1$, such that

$$\frac{\partial\mathscr{L}_i}{\partial v_i}\frac{\partial v_i}{\partial\Omega} = \int_\Omega \{(\boldsymbol{\nabla}\cdot(q\boldsymbol{\nabla}u) + b)\,\delta v_i\}\,\mathrm{d}\boldsymbol{x} = 0, \quad i = 0, 1$$

are equivalent to the condition that u is the solution to the problem (1).

The stationary condition of $\mathscr{L}_i,\ i = 0, 1$ for all variations δu, $\delta u = 0$ on Γ_{D} of u such that

$$\begin{aligned} \frac{\partial\mathscr{L}_0}{\partial u}\frac{\partial u}{\partial\Omega} &= \int_\Omega \{\boldsymbol{\nabla}\cdot(q\boldsymbol{\nabla}v_0) + b\}\,\delta u\mathrm{d}\boldsymbol{x} \\ &\quad - \int_{\Gamma_{\mathrm{N}}} \delta u\,\{p + (q\boldsymbol{\nabla}v_0)\cdot\boldsymbol{n}\}\,\mathrm{d}\gamma \\ &\quad + \int_{\Gamma_{\mathrm{D}}} \{v_0 - u_{\mathrm{D}}\}\,\delta\,(q\boldsymbol{\nabla}u)\cdot\boldsymbol{n}\mathrm{d}\gamma = 0, \\ \frac{\partial\mathscr{L}_1}{\partial u}\frac{\partial u}{\partial\Omega} &= \int_\Omega \{\boldsymbol{\nabla}\cdot(q\boldsymbol{\nabla}v_1)\,\delta u\}\,\mathrm{d}\boldsymbol{x} \\ &\quad - \int_{\Gamma_{\mathrm{N}}} \delta u\,(q\boldsymbol{\nabla}v_1)\cdot\boldsymbol{n}\mathrm{d}\gamma \\ &\quad + \int_{\Gamma_{\mathrm{D}}} v_1\delta\,(q\boldsymbol{\nabla}u)\cdot\boldsymbol{n}\mathrm{d}\gamma = 0 \end{aligned}$$

are equivalent to the condition that $v_i,\ i = 0, 1$ are the solutions to the following adjoint problems, respectively:

$$\begin{aligned} &-\boldsymbol{\nabla}\cdot(q\boldsymbol{\nabla}v_0) = b, \quad \text{in} \quad \Omega,\\ &-(q\boldsymbol{\nabla}v_0)\cdot\boldsymbol{n} = p, \quad \text{on} \quad \Gamma_{\mathrm{N}}, \\ &v_0 = u_{\mathrm{D}}, \quad \text{on} \quad \Gamma_{\mathrm{D}}, \end{aligned} \tag{8}$$

$$\begin{aligned} &-\boldsymbol{\nabla}\cdot(q\boldsymbol{\nabla}v_1) = 0, \quad \text{in} \quad \Omega,\\ &-(q\boldsymbol{\nabla}v_1)\cdot\boldsymbol{n} = 0, \quad \text{on} \quad \Gamma_{\mathrm{N}}, \\ &v_1 = 0, \quad \text{on} \quad \Gamma_{\mathrm{D}}. \end{aligned} \tag{9}$$

Here, we fix u and $v_i,\ i = 0, 1$ with the solutions of problems (1), (8) and (9). By (7), we have the shape gradients for $f_i,\ i = 0, 1$:

$$\frac{\mathrm{d}\mathscr{L}_0}{\mathrm{d}\Omega} = \frac{\partial\mathscr{L}_0}{\partial\Omega} = \{2bu - (q\boldsymbol{\nabla}u)\cdot\boldsymbol{\nabla}u\}\,\boldsymbol{n} = g_0\boldsymbol{n},$$

$$\frac{\mathrm{d}\mathscr{L}_1}{\mathrm{d}\Omega} = \frac{\partial\mathscr{L}_1}{\partial\Omega} = \boldsymbol{n} = g_1\boldsymbol{n}.$$

See Azegami (1994).

4. SOLUTION TO THE PROBLEM (4) USING TIME EVOLUTION EQUATIONS

In order to obtain the solution to the problem (4), we introduce following time evolution equations:

$$\begin{aligned} &\frac{\partial\boldsymbol{v}(t,\boldsymbol{x})}{\partial t} = \boldsymbol{\nabla}\cdot(c\boldsymbol{\nabla}\boldsymbol{v}(t,\boldsymbol{x})) - \alpha\boldsymbol{v}(t,\boldsymbol{x}), \quad \text{in} \quad \Omega_0\times[0,T],\\ &\boldsymbol{v}(0,\boldsymbol{x}) = \boldsymbol{0} \quad \text{in} \quad \Omega_0,\\ &-(c\boldsymbol{\nabla}\boldsymbol{v}(t,\boldsymbol{x}))\cdot\boldsymbol{n} = -(g_0(t) + \lambda_1 g_1(t))\,\boldsymbol{n} \text{ on } \Gamma_{\mathrm{M}}\times[0,T],\\ &\boldsymbol{v}(t,\boldsymbol{x}) = \boldsymbol{0} \quad \text{on} \quad \Gamma_{\mathrm{F}}\times[0,T], \end{aligned} \tag{10}$$

$$\frac{\partial\boldsymbol{\rho}(t,\boldsymbol{x})}{\partial t} = \boldsymbol{v}(t,\boldsymbol{x}) \quad \text{in} \quad [0,T]\times\Omega_0. \tag{11}$$

where $c > 0$, $\alpha \geq 0$ and $T > 0$ are given constants. We chose sufficiently large T so that

$$|f_0(\Omega(T)) - f_0(\Omega(T-\delta t))| \big/ |f_0(\Omega(T-\delta t))| \leq \varepsilon_0$$

holds for a small time step δt and given small constant $\varepsilon_0 > 0$. λ_1 denotes a Lagrange multiplier for f_1. In this paper, we set λ_1 as

$$\lambda_1 = -\frac{\|g_0(t)\boldsymbol{n}\|}{\|g_1(t)\boldsymbol{n}\|}\exp(\beta f_1), \quad \|\cdot\| = \sqrt{\langle\cdot,\cdot\rangle}, \tag{12}$$

so that KKT condition $g_0(T)\boldsymbol{n} + \lambda_1 g_1(T)\boldsymbol{n} = \boldsymbol{0}$, $\lambda_1 f_1(T) = 0$, $\lambda_1 \geq 0$, $f_1(T) \leq 0$ holds at the end of optimization process. Here $\beta > 0$ is the given constant for controlling the violation of constraint function during the optimization process (See Kawamoto (2013)). We consider the violation of f_1 in the optimization process of computation and chose $\beta > 0$ so that $f_1 \leq 0$ holds for all $t \in [0,T]$. The solution to the problem (4) is obtained by $\Omega = \Omega_0 + \boldsymbol{\rho}$.

5. NUMERICAL EXAMPLE

We analyze a two-dimensional problem related to a steady-state heat conduction problem (1). Fig. 1 shows the initial domain Ω_0 with boundary $\partial\Omega_0 = \Gamma_{\mathrm{M}} \sup \Gamma_{\mathrm{F}}$. Fig. 2 shows a design domain Ω with boundary $\partial\Omega = \Gamma_{\mathrm{N}} \cup \Gamma_{\mathrm{D}}$ where $\Gamma_{\mathrm{N}} = \Gamma_{\mathrm{N1}} \cup \Gamma_{\mathrm{N2}}$. Table 1 shows problem settings of the problem (10) and (1). Fig. 3 shows the solution u to the problem (1) in the initial domain Ω_0 and the mesh used in this analysis. Fig. 4 shows the solution u to the problem (1) in the optimized domain Ω. Fig. 5 shows the history of objective function and constraint function during the optimization process. In fig. 5, objective function and constraint function are normalized by using each initial values.

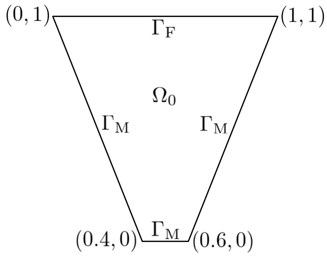

Fig. 1. The original domain Ω_0 with boundary $\partial\Omega_0$

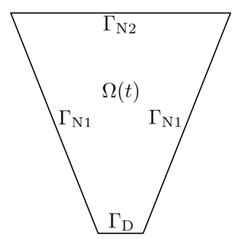

Fig. 2. A design domain Ω with boundary $\partial\Omega$

Table 1. Problem settings of the problem (10) and (1)

Problem	Settings
problem (10)	$T = 1.5$, $c = 1$, $\alpha = 0$, $v_2 = 0$ on $\Gamma_{\mathrm{M}}\times[0,T]$
problem (1)	$q=1$, $b=0$, $p=0$ on Γ_{N1}, $p=-1$ on Γ_{N2}, $u_{\mathrm{D}}=0$
equation (12)	$\beta = 100$

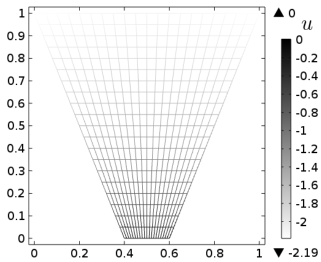

Fig. 3. The solution to the problem (1) in Ω_0

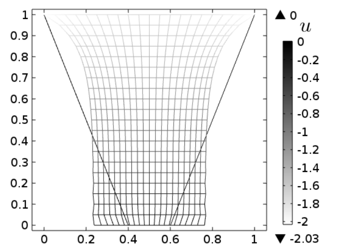

Fig. 4. The solution to the problem (1) in Ω

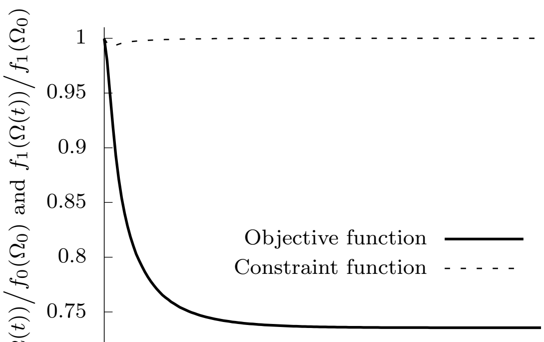

Fig. 5. The history of objective function and constraint function

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