Modeling and analyzing the long-time behavior of random chemostat models ⋆

Tomás Caraballo * **María J. Garrido-Atienza** *
Javier López-de-la-Cruz * **Alain Rapaport** **

* *Dpto. Ecuaciones Diferenciales y Análisis Numérico, Facultad de Matemáticas, Universidad de Sevilla, Calle Tarfia s/n, Sevilla 41012, Spain*
E-mails: {caraball,mgarrido,jlopez78}@us.es.

** *MISTEA (Mathematics, Informatics and Statistics for Environmental and Agronomic Sciences), INRA, Montpellier SupAgro, Univ. Montpellier, 2 place Pierre Viala, 34060 Montpellier cedex 01, France*
E-mail: alain.rapaport@inra.fr

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Chemostat refers to a laboratory device used for growing microorganisms in a cultured environment and has been regarded as an idealization of nature to study competition modeling in mathematical biology, since they can be used to study genetically altered microorganisms, waste water treatment and play an important role in theoretical ecology, to mention a few applications.

The simplest form of chemostat consists of three interconnected tanks called *feed bottle*, *culture vessel* and *collection vessel*, respectively. The substrate or nutrient is pumped from the first tank to the culture vessel and another flow is also pumped from the culture vessel to the third tank such that the volume of the second one remains constant. It leads us to consider the following deterministic chemostat model with Monod kinetics

$$\frac{ds}{dt} = D(s_{in} - s) - \frac{msx}{a+s}, \tag{1}$$

$$\frac{dx}{dt} = -Dx + \frac{msx}{a+s}, \tag{2}$$

where $s(t)$ and $x(t)$ denote concentrations of the nutrient and the microbial biomass, respectively; s_{in} denotes the volumetric dilution rate, a is the half-saturation constant, D is the dilution rate and m is the maximal consumption rate of the nutrient and also the maximal specific growth rate of microorganisms. We notice that all parameters are supposed to be positive and a function Holling type-II, $\mu(s) = ms/(a+s)$, is used as functional response of the microorganisms describing how the nutrient is consumed by the species.

Some standard assumptions are usually imposed when setting up the simplest chemostat model (1)-(2), for instance, it is usually supposed that the availability of the nutrient and its supply rate are both fixed. Nevertheless, this kind of restrictions are really strong since the real world is non-autonomous and stochastic and this is one of the reasons which encourage us to study stochastic and/or random chemostat models.

There are many different ways to introduce stochasticity and/or randomness in some deterministic model, see e.g. Campillo et al. (2011, 2014, 2016); Grasman et al. (2005); Imhof and Walcher (2005); Wang and Jiang (2017); Wang et al. (2016); Xu and Yuan (2015); Zhao and Yuan (2016, 2017). Concerning the chemostat model, the authors in Caraballo et al. (2017a) have already analyzed the simplest chemostat model (1)-(2) in which a stochastic perturbation of the payoff function in continuous-time replicator dynamics is introduced, following the idea developed in Fudenberg and Harris (1992) or in Foster and Young (1990).

Even though there are many different ways to introduce some stochastic perturbation in the chemostat model, it is made on the growth function in most of cases. It could be interesting when the number of individual bacteria is small and there exists some risk of extinction of the biomass in finite time, however this kind of situations hardly ever take place in a nominal regime which is well supervised. Nevertheless, fluctuations on the input flow that brings permanently resources to the bacterial population in continuous cultures are much likely to be observed. In this way, Caraballo and some co-authors already considered a stochastic perturbation on the input flow in the simplest chemostat model (1)-(2) by making use of the standard Wiener process, also called brownian motion or white noise, see e.g. Caraballo et al. (2016) and Caraballo et al. (2017b). Unfortunately, some drawbacks can be found when considering this unbounded stochastic process, for instance, some state variables can take negative values, since the fluctuations could be large enough, what would

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also mean that there is some reverting flow which is a completely unrealistic situation from the biological point of view. In addition, it is not possible to prove the persistence of the microorganisms when perturbing the input flow with the standard Wiener process.

In order to solve the previous problem, another kind of random perturbation on the simplest chemostat model (1)-(2) is analyzed in Caraballo et al. (2017c) by using the well-known Orsntein-Uhlenbeck process, which is a stationary mean-reverting Gaussian stochastic process given by the solution of the Langevin equation

$$dz_t + \beta z_t dt = \nu d\omega_t, \tag{3}$$

where ω_t represents the Wiener process at time $t \geq 0$, $\beta > 0$ is a mean reversion constant that represents how *strongly* our system reacts under some perturbation and $\nu > 0$ is a volatility constant which represents the variation or the size of the noise.

The O-U process can be used to describe the position of some particle by taking into account the friction, which is the main difference with the standard Wiener process and will make our models to be a much better approach to the real ones, in fact, it can be understood as a kind of generalization of the standard Wiener process, which would correspond to take $\beta = 0$ and $\nu = 1$ in (3). Thanks to that, the O-U process consists on a really interesting tool when perturbing the input flow in the chemostat model (1)-(2), since it will allow us to control the perturbations in a suitable way such that the input flow is bounded in some interval to be previously determined by the practitioner.

In conclusion, it can be easily deduce that the O-U process provides us a useful tool to model stochasticity and randomness in the chemostat model since it allows us to lead in models which are a much better approach to the real ones. In addition, this new framework could also be extended to analyze other kinds of models, for instance, those ones with several species and competition and this is currently our main point of interest.

In this work, the simplest chemostat model (1)-(2), perturbing the input flow by means of the O-U process, will be presented. We will make use of the techniques involved in the theory of random dynamical systems (see e.g. Arnold (1998); Caraballo and Han (2016)) to provide some results concerning the existence and uniqueness of global solution just like that the existence and uniqueness of random pullback attractor, which will allow us to obtain detailed information about the long-time behavior of our model. In particular, some conditions on the different parameters of our model will be given to ensure the persistence of the microbial biomass in the *strong* sense

$$\lim_{t \to +\infty} x(t) \geq \rho > 0.$$

Finally, several numerical simulations comparing the results with the ones obtained when perturbing the input flow by using the standard Wiener process will be also shown.

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