On the Periodic Motion of a Two-Body System Upward Along an Inclined Straight Line with Dry Friction ⋆

Tatiana Yu. Figurina *

* *Ishlinsky Institute for Problems in Mechanics RAS, 101-1 Vernadsky Ave, Moscow 119526 Russia, e-mail:t_figurina@mail.ru*

Abstract: The possibility of a two-body system to move upward along an inclined line is investigated. The system is controlled by the force of interaction of the bodies so that the distance between the bodies and their velocities are periodic functions of time. The friction between the bodies and the line is Coulomb's dry friction. Necessary and sufficient conditions for the possibility of periodic upward motion of the system are proved. The motion is possible if and only if the smaller body can start moving upward the line from a state of rest while the bigger body is at rest. An algorithm of the upward motion is presented.

Keywords: limbless locomotion, Coulomb's friction, periodic motion

1. STATEMENT OF THE PROBLEM

Consider a system of two interacting bodies of masses M and m on an inclined plane. Coulomb's dry friction forces act between the bodies and the plane. The force of interaction of the bodies changes the velocities of the bodies, which changes the friction forces that are external forces for the system. Thus, the control of the force of interaction leads to the control of the system's center of mass. The bodies are assumed to move along a fixed line l on an inclined plane. Denote by $\alpha \in [0, \pi/2)$ the angle between line l and the horizontal plane. The bodies are modeled by point masses. Let x and y be the coordinates along line l, and v and V the velocities of bodies m and M, respectively. Without loss of generality we assume $M > m$. Let k be the coefficient of friction against the plane for bodies m and M and g the acceleration due to gravity.

The motion of the system along the line is governed by the equations

$$\begin{aligned} &\dot{x} = v, \qquad \dot{y} = V, \\ &m\dot{v} = -mg\sin\alpha + F + F_m, \\ &M\dot{V} = -Mg\sin\alpha - F + F_M \end{aligned} \tag{1}$$

where F denotes the force applied to body m by body M, and F_m and F_M denote Coulomb's friction forces applied to the bodies by the plane. The friction forces are defined by the relations

$$\begin{aligned} F_m &= -kmg\cos\alpha\,\mathrm{sgn}\,v, \quad v \neq 0, \\ |F_m| &\leq kmg\cos\alpha, \quad v = 0, \\ F_M &= -kMg\cos\alpha\,\mathrm{sgn}\,V, \quad V \neq 0, \\ |F_M| &\leq kMg\cos\alpha, \quad V = 0 \end{aligned} \tag{2}$$

We consider the motions of the system in which the distance between the bodies and the velocities of both bodies are expressed by time-periodic functions, $y(t+T) - x(t+T) \equiv y(t) - x(t)$, $v(t+T) \equiv v(t)$, $V(t+T) \equiv V(t)$. Here, T is the time period which may be chosen arbitrarily. In other words, we consider the motions with constant shifts of each body for the period, the shift being the same for both bodies: $y(t+T) - y(t) \equiv x(t+T) - x(t) \equiv \text{const}$. We call such motions periodic motions. When considering the motions of the system during a period, the periodicity conditions are equivalent to the boundary conditions

$$y(T) - y(0) = x(T) - x(0),\ v(T) = v(0),\ V(T) = V(0) \tag{3}$$

The question is whether the periodic motion of the system upward along the line is possible.

We assume that if there is no interaction between the bodies and both bodies are at rest at some time instant, they will remain at rest on the inclined line. This assumption implies

$$\mathrm{tg}\,\alpha \leq k \tag{4}$$

Additionally, we require that the force governing the uniform upward motion of the smaller body m be not equal in its absolute value to the force governing the uniform downward motion of the larger body M, $kmg\cos\alpha + mg\sin\alpha \neq kMg\cos\alpha - Mg\sin\alpha$. This condition can be written as

$$\mathrm{tg}\,\alpha \neq k\frac{M - m}{M + m} \tag{5}$$

This is necessary for determining uniquely which of the bodies will start moving first from the state of rest of the entire system when an appropriate interaction force begins to act.

Problem. Find the condition allowing the periodic motion for the two-body system upward along the inclined line, provided that relations (1) -(5) and the inequality

$$x(T) > x(0) \tag{6}$$

hold.

2. CRITERIA OF POSSIBILITY OF THE PERIODIC UPWARD MOTION

Proposition. The periodic motion of the two-body system upward along an inclined line is possible if and only if

⋆ This study was supported by the Russian Foundation for Basic Research (projects 17-01-00652, 17-51-12025).

$$\operatorname{tg}\alpha < k\frac{M-m}{M+m} \quad (7)$$

Proof. Let us prove first the sufficiency of the condition (7). To do so, we will construct explicitly a periodic motion of the system that shifts it upward along line l, provided that inequality (7) holds. Let the system be at rest at the beginning of the motion, $v(0) = V(0) = 0$. At the first stage of the motion, body m moves downward, while body M moves upward. Choose some time interval $[0, t_0]$ and a constant interaction force F so that body M begins to move upward, namely, let $F \equiv -(Mg\sin\alpha + kMg\cos\alpha + A)$, $A > 0$. Then body M will move with an acceleration upward and body m will move with an acceleration downward,

$$\begin{aligned} &M\dot{V} = A, \\ &m\dot{v} = -(M+m)g\sin\alpha - (M-m)kg\cos\alpha - A \end{aligned} \quad (8)$$

After that we stop controlling the system and set $F \equiv 0$ for an interval $[t_0, t_1]$, with the duration of this interval being large enough for both bodies to have time to come to a stop due to friction, so that $V(t_1) = v(t_1) = 0$. Denote $x(t_1) = x_1$, $y(t_1) = y_1$, $y_1 > 0$, $x_1 < 0$.

Let at the second stage of the motion body m overtake body M, while body M is at rest. Define the control force F as follows:

$$\begin{aligned} &F(t) = mg\sin\alpha + kmg\cos\alpha + B, \quad t \in [t_1, t_1+\delta], \\ &F(t) = mg\sin\alpha + kmg\cos\alpha - B, \; t \in [t_1+\delta, t_1+2\delta] \end{aligned} \quad (9)$$

If the value of B is small enough, body M stays at rest. Indeed, the inequality (7) implies that there exists a positive value B such that the inequality

$$mg\sin\alpha + kmg\cos\alpha + B + Mg\sin\alpha < kMg\cos\alpha \quad (10)$$

holds. This inequality means that the value of the sliding friction force for body M is greater than the modulus of the sum of all other forces applied to this body; hence body M is at rest. The motion of body m is governed by the equations

$$\begin{aligned} &m\dot{v} = B, \quad t \in [t_1, t_1+\delta], \qquad v(t_1) = 0 \\ &m\dot{v} = -B, \quad t \in [t_1+\delta, t_1+2\delta] \end{aligned} \quad (11)$$

At the time instant $t_1 + 2\delta$, body m comes to a stop, $v(t_1+2\delta) = 0$. By equating the distance travelled by body m for time 2δ to the value $y_1 - x_1$, one can find the duration $2\delta = 2\sqrt{\frac{m}{B}(y_1 - x_1)}$ required for body m to overtake body M and to come to a stop. By letting $T = t_1 + 2\delta$ we complete the construction of the control subject to which the system is at rest at the beginning and at the end of the period and both bodies travel the distance $y_1 > 0$ upward along line l. This completes the proof of sufficiency of the condition (7).

Now we will prove the necessity of (7). Let us suppose that the periodic motion of the system upward along the line is possible. Denote by u the velocity of the center of masses of the system

$$u = (m+M)^{-1}(mv + MV) \quad (12)$$

Equations (1) involve the equation of motion for the center of mass:

$$(m+M)\dot{u} = -(m+M)g\sin\alpha + F_M + F_m \quad (13)$$

If the upward motion of the system is possible, then a time interval exists such that the center of mass velocity is positive on this interval. Hence, one can take a time instant t_* from the left neighborhood of the point of maximum of the function $u(t)$, so that the velocity is positive and its derivative is nonnegative at this point,

$$u(t_*) > 0, \qquad \dot{u}(t_*) \geq 0. \quad (14)$$

The first inequality (14) means that $v(t_*) > 0$ or $V(t_*) > 0$. If $V(t_*) > 0$, then $F_M = -kMg\cos\alpha$ and the second inequality (14) cannot hold by virtue of equation (13) and the inequalities $|F_m| \leq kmg\cos\alpha$ and $m < M$. The center of mass necessarily decelerates if the larger body moves forward. Let now $v(t_*) > 0$. Then $F_m = -kmg\cos\alpha$, and the second inequality (14), with (13) being taken into account, can be represented as follows:

$$F_M \geq (m+M)g\sin\alpha + kmg\cos\alpha \quad (15)$$

Hence, taking into account the relation $|F_M| \leq kMg\cos\alpha$ we obtain $k(M-m)g\cos\alpha \geq (M+m)g\sin\alpha$. This inequality combined with condition (5) lead to (7), which completes the proof of the necessity of the inequality (7).

Remark 1. This proposition can be reformulated as follows. The periodic motion of a two-body system upward along an inclined line is possible if and only if body m can move upward with nonnegative acceleration while body M is at rest. The equivalence of the reformulation to the proposition is proved by the facts that the inequality (7) is equivalent to inequality (10) with B from the right-hand neighborhood of zero and inequality (10) provides the motion of body m upward with nonnegative acceleration with body M at rest.

Remark 2. Let the system of n interacting bodies with masses

$$m_1 \leq m_2 \leq \cdots \leq m_n, \quad M_* = \sum_{i=1,..n} m_i \quad (16)$$

on an inclined line with dry friction be considered. The periodic motion of this system upward along an inclined line is possible if and only if the body with minimal mass m_1 can move upward with a nonnegative acceleration, with all other bodies being at rest. This condition implies

$$\operatorname{tg}\alpha < k\frac{M_* - 2m_1}{M_*} \quad (17)$$

The proof of the necessity of this condition is similar to that for a two-body system. The sufficiency is proved by presenting the motion similar to that presented above, for which at the first stage bodies m_i, $i = 2, \ldots n$ move as a single whole (as the larger body for the two-body case), and at the second stage all these bodies are at rest.

Remark 3. Let now two bodies of the system have different coefficients of friction and let the friction be anisotropic. Denote by k_m^+, k_m^- and k_M^+, k_M^- the coefficients of friction for upward and backward motions of bodies m and M, respectively. We assume that at least one of the inequalities $k_M^- M > k_m^+ m$, $k_m^- m > k_M^+ M$ holds and that the system can stay at rest, which implies

$$\operatorname{tg}\alpha < (k_M^- M + k_m^- m)(M+m)^{-1} \quad (18)$$

The periodic motion of the system upward along an inclined line is possible if and only if

$$\operatorname{tg}\alpha < \frac{\max\{k_M^- M - k_m^+ m,\, k_m^- m - k_M^+ M\}}{M+m} \quad (19)$$

or, which is the same, one body can move upward with nonnegative acceleration while the other body is at rest. The proof of this remark is similar to that of Proposition.