ONE-DIMENSIONAL SIMULATION OF BLOOD FLOW IN ARTERIES USING FINITE ELEMENTS

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Abstract. Diseases of the cardiovascular system are widely spread among the population in developed countries. Therefore research work in this area is of great interest. The scope of this work is to determine and simulate a blood flow model for arteries of the human body. Starting from the Navier-Stokes equations for incompressible fluids, a one-dimensional partial differential equation system containing two state variables is derived. The Finite Element Method is used to solve the system numerically; more precisely a Taylor-Galerkin scheme of second order is applied. To obtain the boundary conditions, the characteristic variables of the partial differential equation system are used.

With the derived model, the blood flow in single arteries as well as at bifurcations of the arterial tree is studied. The system is solved numerically with Matlab. To verify the implementation, several experiments including different pressure functions as test inputs are performed.

The derived model, which describes blood flow in major arteries only, is developed to serve as a part of a dynamical, controlled and identifiable model for the cardiovascular system of the entire human body. This coupled model makes it possible to investigate the influences of physiological changes of the vascular system to the global characteristics of the blood circuit.

1 Introduction

In many developed countries diseases of the cardiovascular system are the most common cause of death, which makes research in that domain particularly important. Among this research work, the modelling and simulation of the cardiovascular system is of great importance. The investigation of the human cardiovascular system can be supported by numerical methods. Due to numerical simulations it is possible to study the effects of pharmaceuticals, develop non-invasive measuring methods or investigate the effects of surgeries before the real surgical operation.

There are different approaches to model and simulate the human cardiovascular system. On one hand, there exist models for the whole cardiovascular system which are easy to handle. The drawback of these models is that they neglect some of the phenomena which actually occur. On the other hand, complex models are used which consider only a part of the cardiovascular system. They are numerically more expensive, but provide more accurate results for the observed section.

This work deals with the modelling and simulation of blood flow in major arteries. Starting from a few assumptions, concerning among others the geometry of the vessel, a one-dimensional model for the blood flow is derived. This model will be integrated in a controlled model for the cardiovascular system of the entire human body.

2 Derivation of the model

The model is based on the Navier-Stokes equations for incompressible fluids:

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\,\boldsymbol{u} + \frac{1}{\rho}\nabla P - \operatorname{div}\left[\nu\left(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T\right)\right] = 0 \tag{1}$$

$$\operatorname{div} \boldsymbol{u} = 0 \tag{2}$$

A cylindrical domain Ω_t of length L is considered in which the Navier-Stokes equations are valid. The domain boundary changes with time t due to the movement of the vessel wall caused by the flow. The vector $\boldsymbol{u}$ describes the flow velocity. Let (x,y,z) denote the cartesian coordinate system. Then, the three components of $\boldsymbol{u}$ describe the flow velocity in x-, y- and z-direction, respectively. Equivalently, the vector $\boldsymbol{u}$ can be written in cylindrical coordinates (r,θ,z). P denotes the pressure, ν the kinematic viscosity and ρ the density of blood.

The following assumptions are made in order get a one-dimensional model for the blood flow:

Axial symmetry: All physical quantities are independent of the angle θ. As a consequence, the cross-section of the vessel is a circle at all time. Thus, the radius of the tube is a function of the position z and the time t.

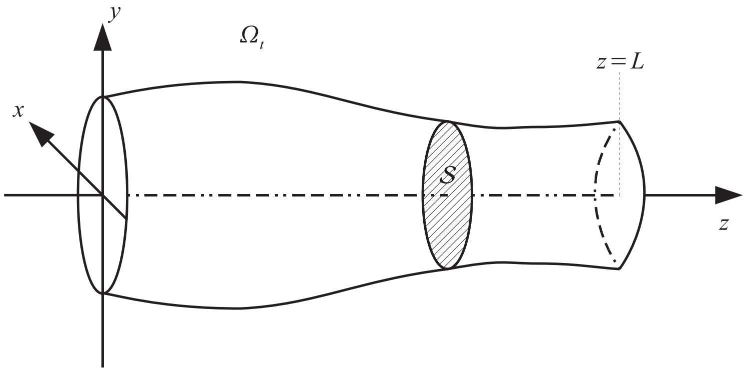


Figure 1: Schematic representation of a vessel segment as cylindrical tube

Radial displacement of the vessel wall: The vessel wall displacement takes place in radial direction only. It can be described by $\eta = R - R_0$, R_0 being a reference radius.

Constant pressure: The pressure is assumed to be constant on every cross-section and thus dependent from z and t only.

Neglect of body forces: Body forces, e.g. gravitation, will be neglected.

Dominance of axial velocity: Since the z-direction is the major direction of blood propagation, the velocity components orthogonal to the z-axis can be neglected. Let u_z denote the velocity component in z-direction. It can be written as follows:

$$u_z(t,r,z) = \bar{u}(t,z)\, s\left(\frac{r}{R(t,z)}\right) \tag{3}$$

where $\bar{u}$ is the averaged velocity on the axial cross-sections and the function $s : \mathbb{R} \to \mathbb{R}$ describes a velocity profile.

Taking into account these assumptions it is possible to reduce the three-dimensional system. In order to get a one-dimensional system, the Navier-Stokes equations are integrated over a generic cross-sectional area $S(z,t)$.
The resulting system consists of two equations with three state variables: the cross-sectional area A, the averaged volume flow Q, and the averaged pressure P. With the help of the following algebraic equation taken from a mechanical model for the vessel wall displacement (see for example [6]), the pressure P can be eliminated from the system at hand.

$$P - P_{ext} = \beta \frac{\sqrt{A} - \sqrt{A_0}}{A_0} \tag{4}$$

In equation (4), β is a linear function of the Young modulus and the thickness of the arterial wall. A_0 denotes the cross-sectional area corresponding to the reference radius R_0.

Thus, the reduced system can be written as follows:

$$\boldsymbol{U} = \begin{pmatrix} A \\ Q \end{pmatrix} \tag{5}$$

$$\frac{\partial \boldsymbol{U}}{\partial t} + \frac{\partial \boldsymbol{F}}{\partial z}(\boldsymbol{U}) = \boldsymbol{B}(\boldsymbol{U}), \quad z \in (0,L),\ t > 0 \tag{6}$$

$$\boldsymbol{F}(\boldsymbol{U}) = \begin{pmatrix} Q \\ \alpha \frac{Q^2}{A} + \frac{\beta}{3\rho A_0} A^{\frac{3}{2}} \end{pmatrix} \tag{7}$$

$$\boldsymbol{B}(\boldsymbol{U}) = \begin{pmatrix} 0 \\ -K_R \frac{Q}{A} - \frac{A}{A_0 \rho}\left(\frac{2}{3}A^{\frac{1}{2}} - A_0^{\frac{1}{2}}\right)\frac{\partial \beta}{\partial z} + \frac{\beta}{\rho}\frac{A}{A_0^2}\left(\frac{2}{3}A^{\frac{1}{2}} - \frac{1}{2}A_0^{\frac{1}{2}}\right)\frac{\partial A_0}{\partial z} \end{pmatrix} \tag{8}$$

where K_R is a friction parameter and α denotes the Coriolis coefficient.

3 Discretization

In order to solve the equations of the one-dimensional model derived in the previous section, a Finite Element Method, more precisely a Taylor-Galerkin scheme of second order, is used, see also [2].

The time axis is subdivided into time steps of length Δt. A Taylor expansion of second order is applied in order to determine the value of $\boldsymbol{U}$ at the next time t^{n+1}.

For the discretization of space, the interval $[0,L]$ is divided into elements $[z_j, z_{j+1}]$. Thus, the length of one element is $h_j = z_{j+1} - z_j$.
Let V_h denote the space of piecewise linear functions. Since a discretization of both variables A and Q needs to be performed, the following spaces are introduced:

$$\boldsymbol{V}_h = V_h^2 \qquad \boldsymbol{V}_h^0 = \{\boldsymbol{v}_h \in \boldsymbol{V}_h |\ \boldsymbol{v}_h = \boldsymbol{0} \text{ for } z = 0,\ z = L\}$$

In this way, the problem can be written in its weak form. Its approximated solution $\boldsymbol{U}_h^{n+1} \in \boldsymbol{V}_h$ must satisfy the boundary conditions and the following equation for all $\varphi_h \in \boldsymbol{V}_h^0$, see [3].

$$\left(\boldsymbol{U}_h^{n+1}, \varphi_h\right) = \left(\boldsymbol{U}_h^n, \varphi_h\right) + \Delta t \left(\boldsymbol{F}_{LW}(\boldsymbol{U}_h^n), \frac{d\varphi_h}{dz}\right) - \frac{\Delta t^2}{2}\left(\boldsymbol{B_U}(\boldsymbol{U}_h^n)\frac{\partial \boldsymbol{F}(\boldsymbol{U}_h^n)}{\partial z}, \varphi_h\right) + \\ + \frac{\Delta t^2}{2}\left(\boldsymbol{H}(\boldsymbol{U}_h^n)\frac{\partial \boldsymbol{F}(\boldsymbol{U}_h^n)}{\partial z}, \frac{d\varphi_h}{dz}\right) + \Delta t \left(\boldsymbol{B}_{LW}(\boldsymbol{U}_h^n), \varphi_h\right) \tag{9}$$

4 Implementation and simulation results

The numerical solution is performed with Matlab. The implementation of the Finite Element Method is done with the help of [1].

The boundary conditions have been neglected in the previous sections. For the numerical solution of the system however, it is necessary to impose boundary conditions at $z = 0$ and $z = L$. Besides that, the considered vessel segment cannot be integrated into the entire cardiovascular system until the choice of appropriate conditions at the boundaries has been made.

Since the section area A and the averaged volume flux Q are the two state variables of the system, it is necessary to provide suitable values at $z = 0$ and $z = L$ for every time step for those variables. However, appropriate data, for instance from measurements, are often not available for the needed boundary values. Hence, the characteristic variables of the system are used to calculate the required boundary conditions.

In the following experiments a pressure function is provided as proximal input. It has the shape of a sinusoidal oscillation for every experiment. The considered arterial segment has a length of 10 cm. The distance between two nodes of the discretization is constant, $h = 0.1\ cm$. The time step Δt is $1 \cdot 10^{-4}\ s$.

In a first experiment, the pulse duration of the input pressure function is 0.165 s. The input pressure is transferred almost unchanged throughout the whole part of the vessel. This behaviour is what can be expected, because it is assumed that there is no pressure at the distal end of the artery. In this way, a tube of infinite length is simulated.

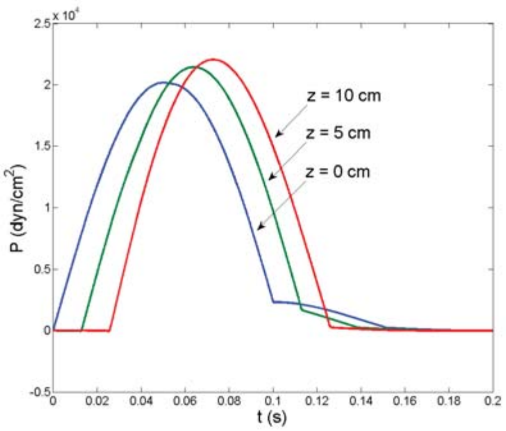
In a second experiment, the behaviour at a bifurcation point will be investigated. Thus, three parts of different arteries, each with a length of 10 cm, are combined in a bifurcation point. The reference radius R_0 is 0.5 cm in the proximal artery, and the reference radii in the two other vessels have a value of 0.4 cm and 0.3 cm, respectively. This time, the duration of the input pulse is 0.1 s. Again, the pressure curve is a value of interest. In the first artery a pressure reflection at the bifurcation can be seen. At $z = 0\ cm$ the pressure pulse is extended because of the backward moving pressure wave. At $z = 10\ cm$ the result of the reflection which takes place exactly at this point is an increased pressure. In the middle of the first vessel a combination of these two effects happens (see figure 2a).

In figure 2b the flow rate in the center of the three arteries ($z = 5\ cm$) is shown over time. One can see that Q is splitting up at the bifurcation point into the two ongoing vessels, whereby more blood is flowing through the larger of the two arteries. In the graph of the flow rate of the proximal artery, a minor negative flow occurs. This is the backflow caused by the reflection at the bifurcation.

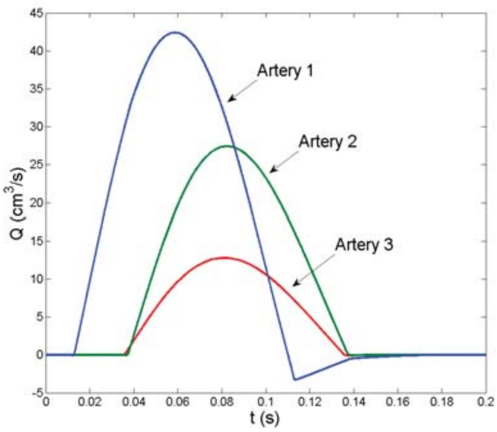
The results described here can be compared with those published in [2] and [3]. They match in a good way.

5 Outlook

With the help of the model described above, blood flow and pressure in a part or in the whole network of larger arteries of the human body can be simulated. For that purpose the vascular system is modelled approximately with straight pipe segments.



(a) Pressure curve at three points of the first artery

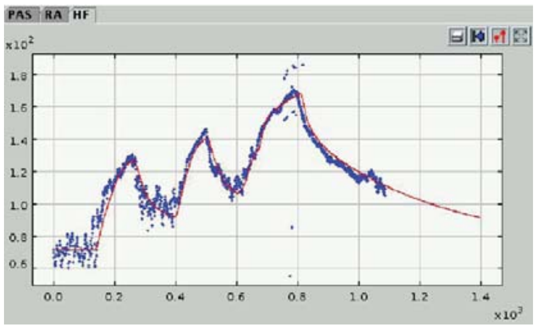


(b) Flow rate at the three centers of the arteries

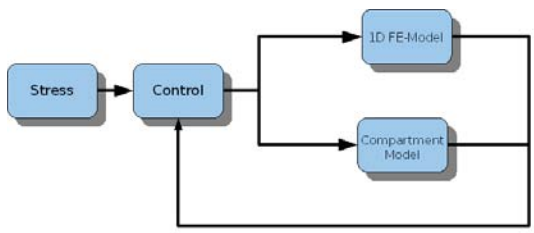
Figure 2: Pressure and flow rate in the arteries

The network model can be coupled with a model for the control of the cardiovascular system (see [4]). In this way, influences like physical strain and their impact on pressure conditions in particular parts of the arterial tree can be studied.

A simulation scheme is illustrated in figure 3b. A difficulty is the multiscale characteristic of the overall model. The output of the controlled model is a value which is averaged over one heart beat, but for the network model a transient input curve (e.g. blood flow in aorta, see figure 3a) is needed (see [5]).



(a) Comparison of simulated and measured data



(b) Scheme of a simulation of the cardiovascular system

Figure 3: Scheme of a simulation of the cardiovascular system

6 References

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