On the determination of effective properties of certain structures with non-periodic temporal oscillations

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Abstract. We investigate the homogenization of an evolution problem modelled by a parabolic equation, where the coefficient describing the structure is periodic in space but may vary in time in a non-periodic way. This is performed applying a generalization of two-scale convergence called λ-scale convergence. We give a result on the characterization of the λ-scale limit of gradients under certain boundedness assumptions. This is then applied to perform the homogenization procedure. It turns out that, under a certain condition on the rate of change of the temporal variations, the effective property of the given structure can be determined the same way as in periodic cases.

1 Introduction

We study an evolution process in a heterogeneous material possible to model by a parabolic equation, e.g. heat conduction. Consider the equation

$$\begin{aligned} \partial_t u^h(x,t) - \nabla\cdot(a(hx,\beta^h(t))\nabla u^h(x,t)) &= f(x,t) \quad \text{in } \Omega_T = \Omega\times(0,T)\,, \\ u^h(x,t) &= 0 \quad \text{on } \partial\Omega\times(0,T), \\ u^h(x,0) &= u^0(x) \quad \text{in } \Omega, \end{aligned} \tag{1}$$

where Ω is an open bounded set in $\mathbb{R}^N$ and the function a is periodic in both arguments with respect to the unit cube Y in $\mathbb{R}^N$ and $(0,1)$, respectively. We assume that the material coefficient

$$a^h(x,t) = a(hx,\beta^h(t)) \tag{2}$$

and hence may develop in a non-periodic fashion in time while the structure is periodic in space with period $\varepsilon = h^{-1}$ for each fixed t and h. Under certain assumptions on a the sequence of solutions u^h will converge (as $h\to\infty$) towards a limit function u, which solves an equation of the same type as above with identical boundary and initial data but with a coefficient b which is constant. This equation is called the homogenized problem. This means that the effective property of a material as described above, for small ε, will be close to that of a material whose properties does not vary in space or time. Problems of this kind, with oscillations in both space and time, have been investigated e.g. in [FlOl1], where the oscillations in space are on several scales, and in [FlOl2], which includes several scales of periodic temporal oscillations. The contribution in this paper concerns the oscillations in time where the periodicity assumption is relaxed.

2 λ-scale convergence

The special case of periodic homogenization, where we have periodic time oscillations according to

$$\beta^h(t) = h^r t, \tag{3}$$

can be treated e.g. utilizing two-scale convergence, see [Ng]. For the more general case (2) we apply a generalization of this concept, which also allow non-periodic structures, called λ-scale convergence, see [HoSi], [Si], [Pe] and [MaTo].

Definition 1 *Let $\{\alpha^h\}$ be a sequence of functions $\alpha^h : \Omega_T \to Y\times(0,1)$. A sequence in $L^2(\Omega_T)$ is said to λ-scale converge to $u_0 \in L^2(\Omega_T\times Y\times(0,1))$ with respect to $\{\alpha^h\}$ if*

$$\lim_{h\to\infty}\int_{\Omega_T} u^h(x,t)v(x,t,\alpha^h(x,t))\,dxdt = \int_{\Omega_T}\int_0^1\int_Y u_0(x,t,y,s)v(x,t,y,s)\,dydsdxdt$$

for all $v\in L^2(\Omega_T;C_\sharp(Y\times(0,1)))$.

A compactness result with respect to this kind of convergence holds if $\alpha^h(x,t)$ is asymptotically uniformly distributed, cf. [HoSi], [Si] and [Pe]. To give a short explanation of this concept we consider functions α^h depending only on x, as originally introduced in [Si]. Let us cover $\alpha^h(\Omega)$ with a collection $\{Y_k\}$ of unit cubes. Moreover, divide each such cube into identical smaller cubes Y_k^j. The asymptotically uniform distribution of α^h means that for any such partition the quotient between the volume of the part of Ω needed for α^h to produce Y_k^j and the volume of the part of Ω used to obtain the entire cube Y_k approaches the portion of Y_k occupied by Y_k^j, i.e. the volume of Y_k^j, when $h \to \infty$. Moreover the radius of $(\alpha^h)^{-1}(Y^j)$ should approach zero uniformly when $h \to \infty$.

The compactness result states that if $\alpha^h(x,t)$ is asymptotically uniformly distributed, then any bounded sequence in $L^2(\Omega_T)$ λ-scale converges up to a subsequence, see [Si], Definition 5, Theorem 6, Definition 10 and Proposition 30. Thus if

$$\alpha^h(x,t) = (hx, \beta^h(t)) \tag{4}$$

is asymptotically uniformly distributed it holds for a subsequence that

$$\lim_{h\to\infty} \int_{\Omega_T} u^h(x,t) a(x,t,hx,\beta^h(t))\, dxdt = \int_{\Omega_T} \int_0^1 \int_Y u_0(x,t,y,s) a(x,t,y,s)\, dydsdxdt \tag{5}$$

for any a in $L^2(\Omega_T; C_\sharp(Y \times (0,1)))$. It is easy to see that α^h given in (4) is asymptotically uniformly distributed if β^h is. Furthermore, if we have strong convergence of $\{u^h\}$ in $L^2(\Omega_T)$ we obtain

$$\lim_{h\to\infty} \int_{\Omega_T} u^h(x,t) a(x,t,hx,\beta^h(t))\, dxdt = \int_{\Omega_T} \int_0^1 \int_Y u(x,t) a(x,t,y,s)\, dydsdxdt, \tag{6}$$

i.e. the local variables y and s vanishes in the λ-scale limit, see Corollary 31 in [Si].

To homogenize (1) we need information about the limit of the gradients of the solutions u^h. Utilizing (5) and (6) we can characterize the λ-scale limit for gradients following the method used in periodic cases, see e.g. [Ng], [Ho], to obtain the following theorem.

Theorem 2 *Let $\{u^h\}$ be a sequence bounded in $H^1(0,T;H_0^1(\Omega),H^{-1}(\Omega))$. Then it holds that*

$$\lim_{h\to\infty} \int_{\Omega_T} \nabla u^h(x,t) a(x,t,hx,\beta^h(t))\, dxdt = \int_{\Omega_T} \int_0^1 \int_Y (\nabla u(x,t) + \nabla_y u_1(x,t,y,s))\, a(x,t,y,s)\, dydsdxdt, \tag{7}$$

where u is the strong $L^2(\Omega_T)$-limit of $\{u^h\}$ and $u_1 \in L^2(\Omega_T \times (0,1); H_\sharp^1(Y)/\mathbb{R})$.

3 Homogenization

In periodic homogenization with β^h given by (3), we may apply two-scale convergence to the weak form of (1) with suitable choices of test functions to arrive at so-called local problems, which make it possible to determine the limit coefficient b. These local problems are of three different types for $0 < r < 2$, $r = 2$ and $r > 2$ respectively. For $0 < r < 2$ the coefficient turns out to have the entries

$$b_{ij} = \int_0^1 \int_Y a_{ij}(y,s) + \sum_{k=1}^N a_{ik}(y,s) \partial_{y_k} z_j(y,s)\, dyds, \tag{8}$$

where z_j is Y-periodic in the first argument and $(0,1)$-periodic in the second and solves the local problem

$$-\nabla_y \cdot (a(y,s)(e_j + \nabla_y z_j(y,s))) = 0 \quad \text{in } Y \times (0,1), \tag{9}$$

see e.g. [BLP] and [Ho].

To perform the homogenization of our problem which may include non-periodic time oscillations we proceed in a similar manner as in the periodic case. Under the assumption that

$$h^{-2} \partial_t \beta^h(t) \to 0 \quad \text{in } L^\infty(0,T), \tag{10}$$

where we assume that $\{\beta^h\}$ is asymptotically uniformly distributed, it turns out that the homogenized problem for the non-periodic case coincides with that of the periodic case (3) for $0 < r < 2$. To see this we make two different choices of test functions in the weak form of (1), that is

$$\int_{\Omega_T} -u^h(x,t)v(x)\partial_t c(t) + a(hx,\beta^h(t))\nabla u^h(x,t)\cdot\nabla v(x)c(t)\,dxdt = \int_{\Omega_T} f(x,t)v(x)c(t)\,dxdt \tag{11}$$

for all $v \in H_0^1(\Omega)$ and $c \in D(0,T)$. First we choose test functions without microscopic variations, i.e. as just mentioned, and pass to the limit using (7) and the strong convergence of $\{u^h\}$ in $L^2(\Omega_T)$, which results in a preliminary stage of the homogenized problem, with b not yet fully characterized;

$$\int_{\Omega_T} -u(x,t)v(x)\partial_t c(t) + \left(\int_0^1\int_Y a(x,t,y,s)\left(\nabla u(x,t)+\nabla_y u_1(x,t,y,s)\right)\,dyds\right)\cdot\nabla v(x)c(t)\,dxdt = \int_{\Omega_T} f(x,t)v(x)c(t)\,dxdt. \tag{12}$$

Next we study problem (11) choosing test functions

$$v(x) = h^{-1}v_1(x)v_2(hx),\ c(t) = c_1(t)c_2(\beta^h(t)),$$

where $v_1 \in D(\Omega)$, $v_2 \in C_\sharp^\infty(Y)/\mathbb{R}$, $c_1 \in D(0,T)$ and $c_2 \in C_\sharp^\infty(0,1)$. We obtain

$$\begin{aligned}&\int_{\Omega_T} u^h(x,t)v_1(x)v_2(hx)\left(h^{-1}\left(\partial_t c_1(t)\right)c_2(\beta^h(t)) + h^{-1}c_1(t)\partial_s c_2(\beta^h(t))\partial_t\beta^h(t)\right) + \\ &a(hx,\beta^h(t))\nabla u^h(x,t)\cdot\left(h^{-1}\nabla v_1(x)v_2(hx)+v_1(x)\nabla_y v_2(hx)\right)c_1(t)c_2(\beta^h(t))\,dxdt = \\ &\int_{\Omega_T} f(x,t)h^{-1}v_1(x)v_2(hx)c_1(t)c_2(\beta^h(t))\,dxdt.\end{aligned} \tag{13}$$

Obviously, the first term vanishes as $h \to \infty$ and so does the right hand side. Due to [AlBr]

$$\int_\Omega hv_1(x)v_2(hx)(\cdot)\,dx$$

is bounded in $H^{-1}(\Omega)$ and hence the second term in the left hand side of (13) goes to zero if

$$h^{-2}\partial_t\beta^h(t) \to 0 \quad \text{in } L^\infty(0,T).$$

Finally, again benefiting from (7) for the remaining part of the left hand side, we arrive at the weak form of the local problem (9) after a separation of local and global variables. Applying the same kind of separation of variables in the homogenized problem (12) attained above we can identify b by means of (8).

Thus it is possible to determine the limit coefficient also in certain non-periodic cases, where the key criterion concerns the rate of change of the function β^h.

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