I. Troch, F. Breitenecker, eds. ISBN 978-3-901608-35-3

A FORMAL METHOD FOR IMPROVING THE TRANSIENT BEHAVIOR OF A NONLINEAR FLEXIBLE LINK

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Abstract. This paper presents a procedure for improving the transient response of a boundary-controlled nonlinear flexible elastic beam via a feed-forward action. It is shown that it is possible to obtain a *formal* analytic solution of an infinite-dimensional linear system, which approximates the nonlinear dynamics, under time-varying boundary conditions in terms of the free vibration modes of the structure, whose calculation is a relatively easy task. The feed-forward action is then obtained by inverting the relationship between input and output variables. A numerical example is presented and some simulation results are discussed.

1 Introduction

Most control systems employ two distinct kinds of control actions: a *feed-back action*, to improve the stability properties of the system and increase its robustness with respect to parameters variations and/or disturbances, and a *feed-forward action*, usually needed to achieve a rapid response and improve tracking performances with respect to a time-varying reference signal. Designing a feed-forward controller requires, generally speaking, to invert (at least partially) the dynamics of the system to be controlled in order to obtain a output-input relationship which is then put in cascade with the open loop system itself.

When dealing with infinite dimensional systems, however, this step is not straightforward, because calculating the direct dynamics of the system in closed form may be a formidable task in itself. Even when the system has already been studied and a solution is available, usually it is developed under some assumptions which are unrealistic in control application, e.g. zero boundary conditions or infinite spatial domain. Conversely, the typical effect of a control system is to introduce time-varying boundary conditions or generate a distributed action across the spatial domain over which the system is defined, and obtain a solution thus becomes much more difficult.

This paper focuses on designing a feed-forward controller for flexible beams. In robotics, this problem has been usually addressed by solving the Lagrange's inverse equation of motion in time domain. In this case, the flexible beam is a finite dimensional system of proper order, obtained under the hypothesis that the position of a point of the link is described by a virtual rigid body motion *plus* a deflection, modeled by an Euler-Bernoulli beam model with "some" zero boundary conditions compatible with the rigid coordinates (e.g. clamp-free, pin-free or pin-pin), [1, 4, 9].

The same problem has been also addressed in the context of flatness-based control, mainly with reference to the Euler-Bernoulli beam model, [8, 15]. On the other hand, the procedure illustrated in this paper is applied to the nonlinear flexible elastic beam model introduced in [12] with time-varying boundary conditions on the position/orientation of the extremities of the link and/or on the applied forces/torques. Upon linearization around the unstressed configuration, this nonlinear model simplifies into the well-known Timoshenko beam model for which a formal expression of the dynamics, i.e. of the solution of the PDE with time-varying boundary conditions, can be computed. This solution is expressed as an infinite series whose convergence properties, differently from [8, 15], are not addressed here and are currently under investigation. Some analogies in the approach can be found in [10] where a similar feed-forward control strategy for the Korteweg-De Vries equation is proposed within the flatness framework. Then, the synthesis of the feed-forward control law (i.e. the time-domain inversion of the equation of motion) is an application of well-known results for linear dynamical systems.

2 Flexible elastic beam model

Consider a slender flexible beam of length L and with an unstressed configuration which is *not* required to be a straight line. If $z \in \mathscr{Z} \equiv [0, L]$ denotes the position along the link in the unstressed configuration, assume that the configuration in the space of the cross section, i.e. the relative configuration between a *body reference* $\mathbb{E}_b(z)$ attached to the cross section and an inertial reference $\mathbb{E}_0$ is given by $h(z) \in SE(3)$ (see also Fig. 1). The distributed port Hamiltonian formulation of the flexible link dynamics is [12]:

$$\frac{\partial}{\partial t}\begin{pmatrix} q \\ p \end{pmatrix} = \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial z} + \begin{pmatrix} 0 & \mathrm{ad}_{(q+\hat{n})} \\ -\mathrm{ad}^*_{(q+\hat{n})} & p\wedge \end{pmatrix} \right\} \begin{pmatrix} \delta_q \mathscr{H} \\ \delta_p \mathscr{H} \end{pmatrix} \tag{1}$$

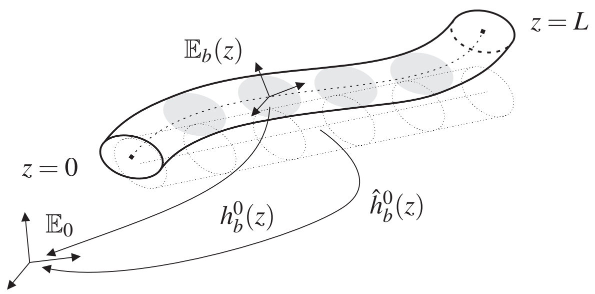


Figure 1: Representation of the nonlinear flexible link in the deformed and unstressed configurations.

with boundary terms given by

$$f^B(0) = \delta_p \mathscr{H}\,|_0 \qquad f^B(L) = \delta_p \mathscr{H}\,|_L \qquad e^B(0) = \delta_q \mathscr{H}\,|_0 \qquad e^B(L) = \delta_q \mathscr{H}\,|_L \tag{2}$$

In (1), $\mathscr{H}$ is the total Hamiltonian, given by the sum of two contributions: the kinetic energy and the potential elastic one due to deformation, i.e.

$$\mathscr{H}(p,q) = \int_{\mathscr{Z}} \left[K(p) + W(q)\right] \mathrm{d}z = \frac{1}{2}\int_{\mathscr{Z}} \left[\langle p \mid p\rangle_Y + \langle q \mid q\rangle_{C^{-1}}\right] \mathrm{d}z \tag{3}$$

Note that the state (energy) variables associated with the flexible link are the infinitesimal deformation q and momentum p, expressed in body reference. If I denotes inertia tensor of the cross section and $p \in se^*(3)$ the momentum of the cross section which corresponds to $t \in se(3)$ via the tensor I, the first contribution in (3) denotes the kinetic energy density, where $Y = I^{-1}$. Note that p is a function of $z \in \mathscr{Z}$ with values in $se^*(3)$. In the same way, C is the compliance tensor, with inverse C^{-1}, which defines a quadratic form on $se(3)$ taking into account the potential elastic energy due to deformation, [18]. Finally, in (1), $\hat{n} = \hat{h}^{-1}\partial_z \hat{h}$ is the "twist" in body reference that provides the direction of the unstressed configuration while δ denotes the variational derivative, [21]. It is clear, then, that the boundary terms in (2) are simply the generalized velocities (i.e. $f^B(\cdot) = \delta_p \mathscr{H}\,|_\cdot$) and forces (i.e. $e^B(\cdot) = \delta_q \mathscr{H}\,|_\cdot$) at the extremities of the beam (i.e. in $z = 0$ and $z = L$).

In case of planar motion, e.g. on the $x-y$ plane of the inertial reference $\mathbb{E}_0$, and under the hypothesis that the unstressed configuration of the beam is a line in the x-direction of the body reference $\mathbb{E}_b$, it is easy to verify that the linearization of (1) results into a transverse wave equation in the displacements in the x-direction, and in the following PDE:

$$\frac{\partial}{\partial t}\begin{pmatrix} p_y \\ p_\theta \\ q_y \\ q_\theta \end{pmatrix} = \begin{pmatrix} 0 & 0 & \partial_z & 0 \\ 0 & 0 & 1 & \partial_z \\ \partial_z & -1 & 0 & 0 \\ 0 & \partial_z & 0 & 0 \end{pmatrix} \begin{pmatrix} \rho^{-1} p_y \\ I^{-1} p_\theta \\ kAG\, q_y \\ EI\, q_\theta \end{pmatrix} \tag{4}$$

in which p_y and p_θ are the translational and rotational momenta of the cross-section, while q_y and q_θ can be identified with the shear and the bending respectively. In [11], it has been proved that (4) is the port Hamiltonian formulation of the Timoshenko beam equation, where ρ is the mass density, E and G are, respectively, the Young's and shear moduli, A is the cross-section area, I is the area moment of the cross-section and k is the shear factor which is used to calculate the transverse shear stiffness of the beam. In the literature, several values for this coefficient have been proposed. This paper uses the derivation of [3], which corresponds (assuming a rectangular cross-section) to a value of $k = 0.85$. If written in terms of the deflection of the beam $w(z,t)$ and of the angle of rotation of the cross-section $\psi(z,t)$, (4) transforms into, [20]:

$$\begin{aligned} \rho A\ddot{w} - kAG\left(w' - \psi\right)' &= 0 \\ \rho I\ddot{\psi} - kAG\left(w' - \psi\right) - EI\psi'' &= 0 \end{aligned} \tag{5}$$

where in the above and later, derivation with respect to time is denoted by a dot and derivation with respect to the spatial variable z by a prime. Finally, let us also define the quantities bending moment $M = -EI\psi'$, transverse shear $Q = kAG(w' - \psi)$ and radius of gyration $r_0 = \sqrt{I/A}$. The feed-forward action for the nonlinear system (1) will be computed on the basis of its linear approximation (5).

3 General solutions of the Timoshenko beam equation

Let us now consider a beam whose dynamics is described by (5). The usual way to deal with time-varying forcing actions is to assume the existence of a solution y in the form $y(t,z) = Z(z)T(t)$, then Laplace transform the PDE to "get rid" of the time derivatives and obtain a ODE in the spatial variable z. Once this equation is solved, its reverse Laplace transform yields the solution sought. This approach has been adopted in [2] and more recently, in the context of feed-forward control design, by [17]. The main limitation of this method is that finding a reverse Laplace transform of an arbitrary function can be quite difficult and moreover, in case the loading condition changes, another inverse transform must be calculated. On the other hand, this paper follows the strategy described in [6, 16] showing that it is possible to obtain formal solutions for forced motions of beams through superposition of the modes of the associated homogeneous problem (i.e. the problem where all boundary conditions are set to zero). Generally speaking, it is applicable under the following assumptions:

- The PDE describing the system dynamics is linear;
- The modes obtained by solving the homogeneous problem form a complete set.

This reduces at calculating the forced response of the system to the solution of an eigenvalue problem (a simple, albeit somewhat tedious task). Moreover, once a given set of modes is known, it is possible to derive quite easily the system dynamics for any input-output combination corresponding to the boundary conditions chosen. Let us assume that external forces may act on the system only through the boundary conditions, that will be expressed by means of the following differential operators:

$$\begin{aligned} B_1(w,\psi) &= w|_{z=0} \text{ or } Q|_{z=0} \\ B_2(w,\psi) &= \psi|_{z=0} \text{ or } M|_{z=0} \\ B_3(w,\psi) &= w|_{z=L} \text{ or } Q|_{z=L} \\ B_4(w,\psi) &= \psi|_{z=L} \text{ or } M|_{z=L} \end{aligned} \tag{6}$$

Once a set of boundary conditions has been chosen among (6) and the initial conditions $w(z,0)$, $\dot{w}(z,0)$, $\psi(z,0)$ and $\dot{\psi}(z,0)$ have been assigned, the solution of (5) exists and is unique.

Let us also assume to have solved the associated homogeneous problem, that is, we know the sequence of eigen-frequencies ω_n and the associated modal shapes $W_n(z)$ and $\Psi_n(z)$ such that

$$w(z,t) = W_n(z)\sin(\omega_n t) \qquad \psi(z,t) = \Psi_n(z)\sin(\omega_n t) \tag{7}$$

are steady-state solutions of (5) when $B_i, i = 1,\dots,4$ are all set to zero. It is also useful to note that the modal shapes satisfy an orthogonality relationship.

Let us now tackle the case of forced vibrations, i.e. $B_i = f_i(t)$ can be time-varying. The solution of this problem will be sought in the form:

$$w(z,t) = \sum_{i=1}^{4} g_{iw}(z) f_i(t) + \zeta_w(z,t) \qquad \psi(z,t) = \sum_{i=1}^{4} g_{i\psi}(z) f_i(t) + \zeta_\psi(z,t) \tag{8}$$

where the functions g_{iw} and $g_{i\psi}$ are determined in such a way to make the boundary conditions on the functions ζ_w and ζ_ψ homogeneous. Such boundary conditions can be written as:

$$B_i(\zeta_w,\zeta_\psi) = f_i(t) - \sum_{j=1}^{4} B_i(g_{jw}, g_{j\psi}) f_j(t),$$

with $i = 1,\dots,4$, from which it is clear to see that requiring homogeneity translates into the following constraint that g_{iw} and $g_{i\psi}$ must satisfy:

$$B_i(g_{jw}, g_{j\psi}) = \delta_{ij}, \qquad i, j = 1,\dots,4 \tag{9}$$

where δ_{ij} is Kronecker's delta. By substituting (8) into (5) we obtain an homogeneous problem in ζ_w and ζ_ψ: whose solution will be sought in the form:

$$\zeta_w(z,t) = \sum_{n=1}^{\infty} W_n(z) T_n(t) \qquad \zeta_\psi(z,t) = \sum_{n=1}^{\infty} \Psi_n(z) T_n(t) \tag{10}$$

where the T_n are the solutions of the following sequence of ODEs:

$$\ddot{T}_n + \omega_n^2 T_n = -\sum_{i=1}^{4} G_{in} \ddot{f}_i + \sum_{i=1}^{4} G_{in}^* f_i = F_n(t) \tag{11}$$

where:

$$G_{in} = \frac{\displaystyle\int_0^L \left(g_{iw} W_n + r_0^2 g_{i\psi} \Psi_n\right) \mathrm{d}z}{\displaystyle\int_0^L \left(W_n^2 + r_0^2 \Psi_n^2\right) \mathrm{d}z} \tag{12}$$

$$G_{in}^* = \frac{\displaystyle\int_0^L \left\{ \frac{kG}{\rho}(g_{iw}'' - g_{i\psi}') W_n + r_0^2 \left[\frac{E}{\rho} g_{i\psi}'' + \frac{kG}{\rho r_0^2}(g_{iw}' - g_{i\psi}) \right] \Psi_n \right\} \mathrm{d}z}{\displaystyle\int_0^L \left(W_n^2 + r_0^2 \Psi_n^2\right) \mathrm{d}z} \tag{13}$$

The solution of (11) is given by:

$$T_n(t) = \left[P_n - \sum_{i=1}^{4} G_{in} f_i(0)\right] \cos(\omega_n t) + \frac{1}{\omega_n}\left[V_n - \sum_{i=1}^{4} G_{in} \dot{f}(0)\right] \sin(\omega_n t) + \frac{1}{\omega_n}\int_0^t F_n(\tau)\sin[\omega_n(t-\tau)]\,\mathrm{d}\tau \tag{14}$$

where the coefficients P_n and V_n are determined, respectively, by the initial conditions on position and velocity. This completes the formal solution of the problem. Note that the problem of finding conditions under which (10) converges is still open. On the other hand, the following section will show how these results can be used to obtain an input-output relationship that can be exploited to design a feed-forward controller.

4 Feed-forward action for the rotating Timoshenko beam

Let us now focus on a more specific example, a beam which is constrained to rotate thanks to a pin joint located at $z = 0$. Let that point be the origin of an horizontal plane where the motion takes place and let the forcing action be a torque $\tau_q(t)$ applied to the joint. To use such a model, it is necessary to devise a way to include the rigid-body mode of rotation into the beam model. One of the simplest way is to define θ as the weighted average angle of inclination through the origin (to be defined more precisely by (17)) and to assume that any longitudinal elongation and stress of the beam due to rotation is small and may be neglected.

By denoting with (z, w) the position vector of a point of the center-line of the beam at distance z from the origin, expressed in the moving frame associated with the beam, and with $(\tilde{w}, \tilde{z})$ the corresponding quantities in the fixed reference frame, it has been proved in [19] that the dynamic model expressed in the fixed frame has the same form of (5), provided that we set:

$$\tilde{w} = w + z\theta \qquad \dot{\tilde{w}}^2 = (\dot{w} + z\dot{\theta})^2 \approx \dot{w}^2 + z^2\dot{\theta}^2 \qquad \tilde{\psi} = \psi + \theta \qquad \dot{\tilde{\psi}}^2 = (\dot{\psi} + \dot{\theta})^2 \approx \dot{\psi}^2 + \dot{\theta}^2 \tag{15}$$

and the rigid-body mode coupling equation is given by:

$$\underbrace{\left[\int_0^L \rho I + \rho A z^2 \mathrm{d}z\right]}_{J_\theta} \ddot{\theta} = \tau_q \tag{16}$$

where θ is defined by:

$$\theta = \frac{\displaystyle\int_0^L (\rho I \tilde{\psi} + \rho A z \tilde{w})\,\mathrm{d}z}{\displaystyle\int_0^L \left(\rho I + \rho A z^2\right)\mathrm{d}z} \tag{17}$$

It is now possible to work out the solution of the rotating beam problem: let the input signal of our system be $u(t) = \tau_q(t)$ and the output to be controlled the tip position, that is $y(t) = \tilde{w}(L,t)$. The prescribed boundary conditions are:

$$B_1 = \tilde{w}(0) = 0 \qquad B_2 = M(0) = u(t) \qquad B_3 = Q(L) = 0 \qquad B_4 = M(L) = 0 \tag{18}$$

Once the boundary condition are set, it is possible to try and calculate the g_{iw} and $g_{i\psi}$. In general, there are multiple ways to satisfy the conditions (9), but a simple way to determine these functions is to assume they are polynomials with unknown coefficients that must be derived by imposing (9). In this case, only g_{2w} and $g_{2\psi}$ are needed and a solution is found to be:

$$g_{2w}(z) = 0 \qquad\qquad g_{2\psi}(z) = \frac{z^2}{2EIL} - \frac{z}{EI} + \frac{L}{2EI} \tag{19}$$

The next step deals with the solution of the eigenvalue problem in order to find the numerical values of the resonant frequencies ω_n and the expressions of the modal shapes W_n and Ψ_n. To avoid dealing with cumbersome coefficients and expressions, we shall use in the following passages the parametrization of [7], as reported in Table 1. By assuming that the solution of the homogeneous problem is (7), it is possible to obtain a set of ODEs whose solution is not identically zero only for some values of the frequency ω_n. After a lengthy but straightforward calculation, it can be shown that such frequencies can be obtained as solutions of the equation:

$$(\alpha^2 + \beta^2)\left[\frac{\beta^2 - s^2}{\alpha}\cosh(b\alpha)\sin(b\beta) - \frac{\alpha^2 + s^2}{\beta}\sinh(b\alpha)\cos(b\beta)\right] = 0 \tag{20}$$

if $\sqrt{(r^2 - s^2)^2 + 4/b^2} > (r^2 + s^2)$, and

$$(\beta^2 - \alpha'^2)\left[\frac{s^2 - \beta^2}{\alpha'}\cos(b\alpha)\sin(b\beta) - -\frac{s^2 - \alpha'^2}{\beta}\sin(b\alpha')\cos(b\beta)\right] = 0 \tag{21}$$

Table 1: Variable correspondence for the eigenvalue problem, [7].

variable	value
b	$\sqrt{\frac{\rho}{EI}}L^2\omega_n$
r^2	$\frac{I_\rho}{\rho L^2}$
s^2	$\frac{EI}{KL^2}$
α	$\frac{1}{\sqrt{2}}\sqrt{-(r^2+s^2)+\sqrt{(r^2-s^2)^2+\frac{4}{b^2}}}$
α'	$j\alpha$
β	$\frac{1}{\sqrt{2}}\sqrt{(r^2+s^2)+\sqrt{(r^2-s^2)^2+\frac{4}{b^2}}}$

otherwise. The modal shapes associated to the Timoshenko beam are in the form:

$$\begin{aligned} W_n(x) &= C_1\cosh(b\alpha\xi)+C_2\sinh(b\alpha\xi)+C_3\cos(b\beta\xi)+C_4\sin(b\beta\xi) \\ \Psi_n(x) &= C_1'\sinh(b\alpha\xi)+C_2'\cosh(b\alpha\xi)+C_3'\sin(b\beta\xi)+C_4'\cos(b\beta\xi) \end{aligned} \tag{22}$$

if $\sqrt{(r^2-s^2)^2+4/b^2}>(r^2+s^2)$, and

$$\begin{aligned} W_n(x) &= C_1\cos(b\alpha'\xi)+jC_2\sin(b\alpha'\xi)+C_3\cos(b\beta\xi)+C_4\sin(b\beta\xi) \\ \Psi_n(x) &= jC_1'\sin(b\alpha'\xi)+C_2'\cos(b\alpha'\xi)+C_3'\sin(b\beta\xi)+C_4'\cos(b\beta\xi) \end{aligned} \tag{23}$$

otherwise. Of the eight coefficients, only half are independent: that is, once C_1, C_2, C_3 and C_4 are set, the other four are uniquely determined (and vice versa). The correspondence is given by:

$$C_1'=\frac{b}{L}\frac{\alpha^2+s^2}{\alpha}C_1 \qquad C_2'=\frac{b}{L}\frac{\alpha^2+s^2}{\alpha}C_2 \qquad C_3'=-\frac{b}{L}\frac{\beta^2-s^2}{\beta}C_3 \qquad C_4'=\frac{b}{L}\frac{\beta^2-s^2}{\beta}C_4$$

if $\sqrt{(r^2-s^2)^2+4/b^2}>(r^2+s^2)$, and

$$C_1'=-j\frac{b}{L}\frac{s^2-\alpha'^2}{\alpha'}C_1 \qquad C_2'=-j\frac{b}{L}\frac{s^2-\alpha'^2}{\alpha'}C_2 \qquad C_3'=-\frac{b}{L}\frac{\beta^2-s^2}{\beta}C_3 \qquad C_4'=\frac{b}{L}\frac{\beta^2-s^2}{\beta}C_4$$

otherwise. The actual value of the coefficients appearing in (22), (23) is determined by the mode number and the boundary conditions and in any case is defined up to a multiplicative factor. If we set C_1 to be this multiplicative factor, their value is found to be:

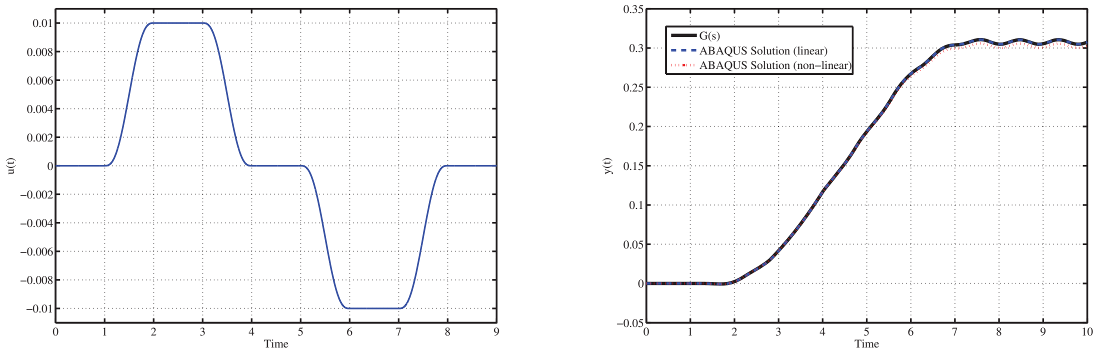
$$\begin{aligned} C_4 &= \frac{C_1}{\beta\chi_\beta-\alpha\chi_\alpha}\Big[\beta(\beta^2-s^2)\cosh(b\alpha)\cos(b\beta)+\beta(\alpha^2+s^2)+\alpha(\alpha^2+s^2)\sinh(b\alpha)\sin(b\beta)\Big] \\ C_3 &= -C_1 \\ C_2 &= \frac{\alpha\cos(b\beta)C_4+[\alpha\sin(b\beta)-\beta\sinh(b\alpha)]C_1}{\beta\cosh(b\alpha)} \end{aligned}$$

if $\sqrt{(r^2-s^2)^2+4/b^2}>(r^2+s^2)$, and

$$\begin{aligned} C_4 &= \frac{C_1}{\alpha'\chi_\alpha'+\beta\chi_\beta'}\Big[\beta(\beta^2-s^2)\cos(b\alpha')\cos(b\beta)+\beta(s^2-\alpha'^2)-\alpha'(s^2-\alpha'^2)\sin(b\alpha')\sin(b\beta)]\Big] \\ C_3 &= -C_1 \\ C_2 &= j\frac{\alpha'\cos(b\beta)C_4+[\alpha'\sin(b\beta)-\beta\sin(b\alpha')]C_1}{\beta\cosh(b\alpha')} \end{aligned}$$

otherwise, where:

$$\chi_\alpha=(\alpha^2+s^2)\sinh(b\alpha)\cos(b\beta) \qquad \chi_\beta=(\beta^2-s^2)\cosh(b\alpha)\sin(b\beta)$$
$$\chi_\alpha'=(s^2-\alpha'^2)\sin(b\alpha')\cos(b\beta) \qquad \chi_\beta'=(\beta^2-s^2)\cos(b\alpha')\sin(b\beta)$$



(a) Input signal $u(t)$.

(b) The beam response $y(t)$.

Figure 2: Comparison between the response provided by $\tilde{G}(s)$ and the FE models.

Now, by combining (8), (10) and (14) and by assuming zero initial conditions and that at $t = 0$ no forcing action is present, in light of (15) and (18), a particularly simple expression is obtained for $y(t) = \tilde{w}(L,t)$:

$$y(t) = L\theta + \sum_{n=1}^{\infty} \frac{W_n(L)}{\omega_n} \int_0^t (-G_{2n}\ddot{u}(\tau) + G_{2n}^{*}u(\tau)) \sin[\omega_n(t-\tau)]\mathrm{d}\tau \tag{24}$$

By recalling (16) and applying the Laplace transform to the whole expression, we finally get:

$$Y(s) = \underbrace{\left\{ \frac{L}{J_\theta s^2} + \sum_{n=1}^{\infty} W_n(L) \frac{G_{2n}^{*} - G_{2n}s^2}{s^2 + \omega_n^2} \right\}}_{G(s)} U(s) \tag{25}$$

where $G(s)$ is the expression of the infinite-dimensional dynamics that links $y(t)$ to $u(t)$. It not straightforward to prove the convergence of the series but, in practice, any implementation will truncate the number of terms in the series to N_m, and then perform the dynamic inversion (that is, calculate $G^{-1}(s)$) on this finite-dimensional approximation $\tilde{G}(s)$. If the control action is such that bandwidth of the input stays well below ω_{N_m}, however, we can treat this $\tilde{G}(s)$ as analytically exact up to the limits of the infinite-dimensional models (Euler-Bernoulli or Timoshenko) used to derive it.

The accuracy of the approximated solution obtained by truncating the terms in (25) to N_m has been analyzed with reference to a beam with unitary length, with $E = 200e06$, $G = 75e06$ and $\rho = 7800$, and with square cross-section of side $0.01m$. A model of this rotating beam was developed using ABAQUS©, a software package used for finite-element (FE) analysis The beam geometry was meshed using 100 B22 elements, i.e. 3-node planar beam elements with quadratic interpolation derived using Timoshenko's theory. The number of the elements used is overwhelmingly more than the strict necessary, but the objective here is to obtain a *virtual prototype* of the beam, that can be used to validate the finite-dimensional approximation $\tilde{G}(s)$.

The finite element model so obtained underwent a three step analysis: a first step of frequency extraction in order to make sure that the natural frequencies of the beam matched those obtained by the frequency equations (20) and (21), followed by two steps of dynamic simulation using the implicit solver ABAQUS/Standard to observe the beam response to a specified control input (see Fig. 2(a)). The dynamic analysis was performed first by suppressing nonlinear effects due to large deformations in order to simulate exactly a Timoshenko beam model (which is based on a linearization of the equation of the elasticity), and then by including them. At the same time, the transfer function of the beam (25) was approximated by setting $N_m = 4$ and the response under the same control input was computed. A comparison of the results obtained is presented in Fig. 2(b). It can be seen that $\tilde{G}(s)$ and the linearized FE model yield practically identical responses, as expected; the behavior of the full non-linear model is instead slightly different, and this is due both to the limitation of the Timoshenko model itself and to the approximations (15) that were used to introduce the rigid body mode.

5 Control design and simulation results

In this section, a possible solution for the synthesis of the feed-forward control action that takes advantage from the formal solution of the equation of motions computed in the previous section is presented, together with some simulation results that prove the practical validity of the approach. The design of the control unit relies on the inversion of the finite-dimensional approximation (25). This step becomes more complicated as the transfer function so obtained is non-minimum phase. This is a common characteristic of the beam models that assume (18) as

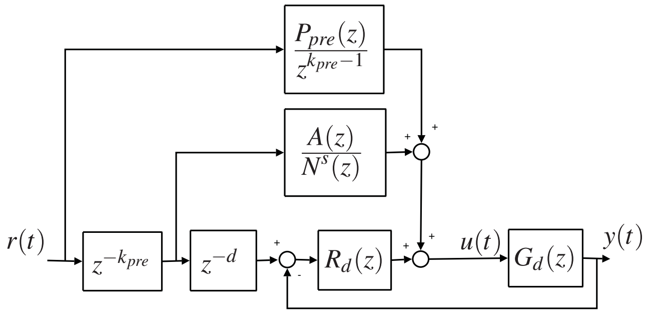


Figure 3: The control setup.

boundary conditions. Generally speaking, this means that the compensator will be only an approximation of the inverse dynamics, that would be otherwise unstable.

It is outside the scope of this paper to review all the techniques investigated in the literature to compute a non-causal inverse of a linear system and so we shall focus on a specific solution proposed by [13] which is based on the so-called *steering along zeros control (SAZC)* strategy by [14]. This method assumes the plant to be controlled to be a discrete-time system, but this is not particularly restrictive. Since most implementations of control units are digital, this simply means approaching the synthesis problem by directly working on a discretized model of the system rather than designing a continuous-time regulator that must be discretized later.

Let us denote by $G_d(z) = N(z)/D(z)$ the discrete-time model of the system, obtained in this example by using a matched pole/zero discretization method with sampling time of $0.005s$. Furthermore, let $N(z) = N^u(z)N^s(z)$, where $N^u(z)$ is made up by the unstable zeros of $G_d(z)$, whereas $N^s(z)$ contains the stable ones. Note that this is equivalent to require that $G_d(z)$ has no zeros on the unit circle. This is a rather strong assumption that will be discussed later.

The control scheme is shown in Fig. 3. As it can be seen, the feed-forward action is decomposed into two contributions: the first one, $A(z)/N^s(z)$, is the standard inversion of the minimum-phase dynamics whereas the second one, $P_{pre}(z)/z^{k_{pre}-1}$, is designed to steer the state of the system (for $t < t_0$, the instant when the reference motion starts) towards a suitable initial state which guarantees perfect tracking with bounded control signals. In practice, only an approximate inverse can be computed as the exact solution of this problem would require infinite pre-action time, so the convolution of unstable zeros in the reverse time direction is truncated after a finite number of samples k_{pre}. The delay z^{-d} takes into account the relative degree d of $G_d(z)$ and the feed-back regulator $R_d(z)$ is necessary to provide the necessary robustness to the system in case of external disturbances and not perfect knowledge of the parameters of the link, which is *always* the case in practical applications.

The unknown polynomials in Fig. 3 can be determined, once k_{pre} has been chosen, by solving two Diophantine equations:

$$\begin{aligned} A(z)N^u(z) + B(z)N^s(z) &= D(z) \\ B_0(z)z^{k_{pre}} + zP_{pre}(z)N^u(z) &= B(z) \end{aligned} \tag{26}$$

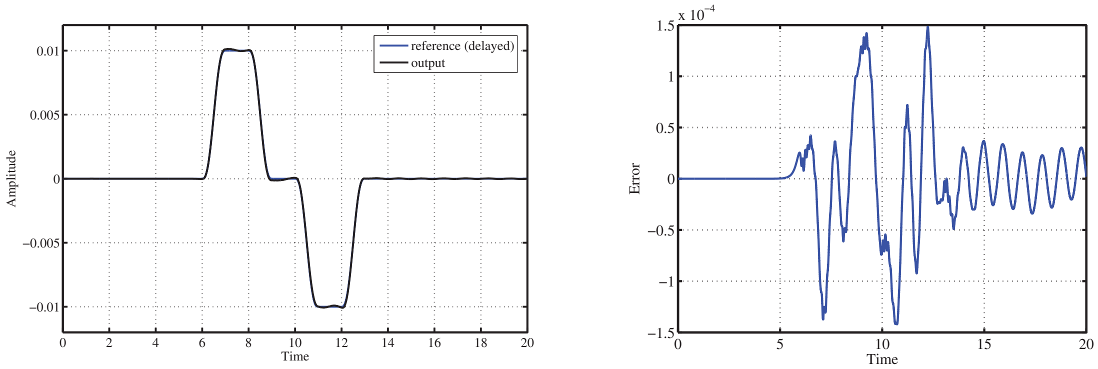
Note that (26) are correct if $N(z)$ is monic, otherwise an additional gain must be incorporated in the controller.

Even if this kind of controller is quite easy to design, its working principle requires the zero-dynamics to be asymptotically converging to zero, either in forward or reverse time. There is nothing, however, preventing $G_d(z)$ to have zeros on the unit circle and, in fact, the transfer function of this example exhibits this feature. A simple yet effective way around this problem is to consider a "perturbed" output $\hat{y}(k)$, defined as:

$$\hat{y}(k) = G_d(z)u(k) + \varepsilon u(k) = \underbrace{\frac{N(z) + \varepsilon D(z)}{D(z)}}_{\hat{G}_d(z)} \tag{27}$$

where ε is assumed to be small enough such that $\hat{y}(k) \approx y(k)$, yet sufficient to move the zeros of the perturbed transfer function outside the unit circle. If this (weaker) assumption is satisfied, the method described above may still be applied to $\hat{G}_d(z)$, and the resulting controller used with $G_d(z)$. Actually, by using a more general feed-through compensator and putting $\hat{y}$ in negative feedback with the control input, this strategy could also be used to derive a stabilizing controller for the system, as discussed in [5].

It is arguable that the control problem examined until now is somewhat unrealistic, as pure open-loop control of such a system would be possible in simulation only. Moreover, as it turned out, even then the performances could not be up to the expectations, if "enough pathologies" arise. A more complete application would of course include also the feed-back controller $R_d(z)$ (see Fig. 3), in order to correct the approximations of the feed-forward



(a) The beam response $y(t)$

(b) Tracking error

Figure 4: Tracking performances with feed-back and feed-forward units.

action and provide robustness to the system. A final simulation has therefore been made by including a digital PD controller with low gains. Using an proportional gain of 0.55 and a derivative gain of 0.001, the results are plotted in Fig. 4(a) and 4(b). Now the feed-back action intervenes by eliminating the drift that could be seen in the previous simulation and reduces the overall magnitude of the tracking error.

6 Conclusions

In this paper, a procedure to determine a feed-forward controller for a nonlinear flexible link model based on its linearization around the unstressed configurations which turns out to be the Timoshenko beam model. Its key feature is to bring a potentially difficult problem, such as determining an analytic input-output relationship in presence of time-varying boundary conditions and distributed forcing actions, within the well-known framework of modal analysis and polynomial transfer functions. The analytic solution in turn provides a complete understanding of the assumptions that can be made without neglecting relevant dynamics and enables the re-use of a variety of techniques coming from linear system theory on the resulting approximate model. A simple, yet significant example of the use of such method is presented in the second half of the paper, together with a numerical example and simulation results.

7 References

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