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HAMILTONIAN FIELD THEORY AND MECHANICS

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Abstract. We analyze the geometry of the evolutionary and the polysymplectic approach in first order Hamiltonian field theory. The main difference important for the application is beside a different bundle construction the different Legendre transform as well as the analysis of the conserved quantities. These theoretical investigations will be completed by the analysis of continuum mechanics in the presented framework.

1 Introduction

The Hamiltonian formalism is well known describing phenomena which can be modeled by ordinary differential equations. The main ingredients of the theory are a representation of the equations in an evolutionary first order form and under some regularity assumptions the Legendre transform yields a connection with the well known Lagrangian description. There exist several approaches that extend this Hamiltonian description to systems that are described by partial differential equations, where the general question arises if the formulation as an evolutionary description should be maintained. A description in evolutionary form requires to single out an evolution parameter, for example the time and this leads to the introduction of the variational derivative with respect to the Hamiltonian. This approach is well known in the literature, see for example [5] or [7] and references therein. Another possibility to describe field theory is an approach going back to *De Donder* which is based on the conservation of the symmetry with respect to all the independent variables, i.e. the time and the space, which leads to the introduction of so called multimomenta and of course the Legendre transform differs from the case of the evolutionary approach. There exists an extensive literature describing the polysymplectic or the multisymplectic formulation, see for example [1, 2, 3]. This paper aims to give a geometric description of both of these approaches in a geometric fashion, singling out the differences in the bundle construction and to show how this two approaches differ in the case of first order field theory in mechanics. As an example we will treat continuum mechanics in great detail, where we assume that a stored energy function exists.

2 Technical Preliminaries

This section includes the main notions of differential geometry and specifies the tensor notation which will be used in this sequel. In this contribution we use the concept of bundles [1] and [6]. Given a bundle $\rho : \mathscr{E} \to \mathscr{X}$ we are able to derive the first jet manifold $\mathscr{J}^1(\mathscr{E})$, with coordinates $(x^i, y^\alpha, y^\alpha_i)$ and the first jet of a section $\sigma : \mathscr{X} \to \mathscr{E}$ is written as $j^1(\sigma)$. The tangent bundle $\mathscr{T}(\mathscr{E})$ possesses the induced coordinates $(x^i, y^\alpha, \dot{x}^i, \dot{y}^\alpha)$ and the vertical subbundle $\mathscr{V}(\mathscr{E})$ is equipped with the coordinates $(x^i, y^\alpha, \dot{y}^\alpha)$, whereas the cotangent bundle $\mathscr{T}^*(\mathscr{E})$ possesses the induced coordinates $(x^i, y^\alpha, \dot{x}_i, \dot{y}_\alpha)$. The special vector field $d_i = \partial_i + y^\alpha_i \partial_\alpha \in \mathscr{T}(\mathscr{E})$, we omit the pullback bundle structure here, that meets the relation $\left(j^1\sigma\right)^* d_i(g) = \partial_i(g \circ \sigma)$ with $g \in \mathscr{C}^\infty(\mathscr{E})$ and $\sigma \in \mathscr{X} \to \mathscr{E}$ is called the first order total derivative. A connection on the bundle $\mathscr{E} \to \mathscr{X}$ is regarded as the map $\Gamma : \mathscr{E} \to \mathscr{J}^1(\mathscr{E})$ which can be represented as

$$\Gamma = \mathrm{d}x^i \otimes (\partial_i + \Gamma^\alpha_i \partial_\alpha), \quad \Gamma^\alpha_i \in \mathscr{C}^\infty(\mathscr{E}). \tag{1}$$

We us the standard notation for tensor bundles as well as for the exterior algebra concerning differential forms, where the interested reader is referred to [1] for a detailed exposition. For the Lie derivative of a geometric object Δ with respect to a vector field v we arrange the notation $v(\Delta) = L_v(\Delta)$. For a tensor field $t_{\alpha\beta}\mathrm{d}x^\alpha \otimes \mathrm{d}x^\beta$ we introduce $t^{\alpha\beta}\partial_\alpha \otimes \partial_\beta$, such that $t_{\alpha\beta}t^{\rho\alpha} = \delta^\rho_\beta$ is met, if it exists, with the Kronecker delta δ.

3 The polysymplectic Structure

The aim of this part is to describe the main concepts of the polysymplectic formalism for first order field theory. Most of the material presented here can be found in [1, 3] but in the latter part of this section we will focus then on the conserved quantities to give a connection to the evolutionary point of view. Let us consider the bundle $\mathscr{E} \to \mathscr{X}$ possessing the coordinates $(x^i, y^\alpha) \to (x^i)$. In our application, continuum mechanics, the coordinates y^α will be denoted by q^α and correspond to the spatial coordinates in the configuration manifold, whereas the independent coordinates x^i will be the spatial coordinates in the reference manifold X^j as well as the time $t^0 = x^0$. Therefor we have $\dim(\mathscr{X}) = n = s+1$ and $i = 0, \ldots, n$. Let us consider the first jet bundle $\mathscr{J}^1(\mathscr{E}) \to \mathscr{E}$, which is affine and a first order Lagrangian L

$$L : \mathscr{J}^1(\mathscr{E}) \to \overset{n}{\wedge}\mathscr{T}^*(\mathscr{X}) \tag{2}$$

$L = \mathscr{L}\omega$ with $\mathscr{L} \in C^\infty\left(\mathscr{J}^1(\mathscr{E})\right)$ together with

$$\omega = \mathrm{d}x^0 \wedge \ldots \wedge \mathrm{d}x^n\ ,\ \ \omega_i = \partial_i \rfloor \omega\,.$$

The partial differential equations for a first order Lagrangian follow as

$$\delta_\alpha(\mathscr{L}) = 0\ ,\ \ \delta_\alpha = \partial_\alpha - d_i\partial_\alpha^i \tag{3}$$

which is a well known result in variational calculus, see for example [1, 6, 5] and references therein. We do not discuss boundary terms here in the Lagrangian context, but they follow in a geometric fashion by the usual variational principle and the theorem of Stokes.

A possible choice for a Lepage equivalent for L is the Poincare Cartan form

$$H_L = \mathscr{L}\omega + \partial_\alpha^i\mathscr{L}(\mathrm{d}y^\alpha - y_k^\alpha \mathrm{d}x^k) \wedge \omega_i = \partial_\alpha^i\mathscr{L}\mathrm{d}y^\alpha \wedge \omega_i - \left(\partial_\alpha^i\mathscr{L}y_i^\alpha - \mathscr{L}\right)\omega\,. \tag{4}$$

Let us consider the Legendre bundle [1]

$$\Pi \to \mathscr{E}\ ,\ \ \Pi = \overset{n}{\wedge}\mathscr{T}^*(\mathscr{X}) \otimes \mathscr{V}^*(\mathscr{E}) \otimes \mathscr{T}(\mathscr{X})$$

with coordinates $(x^i, y^\alpha, p_\alpha^i)$ for Π which possesses the transitions functions

$$\bar{p}_{\bar{\alpha}}^{\bar{i}} = \det(\partial_{\bar{i}}\hat{\phi}^i)\partial_{\bar{\alpha}}\hat{\varphi}^\alpha \partial_i \phi^{\bar{i}} p_\alpha^i$$

with respect to a bundle morphism $\bar{x}^{\bar{i}} = \phi^{\bar{i}}(x)$ and $\bar{y}^{\bar{\alpha}} = \varphi^{\bar{\alpha}}(x,y)$. With the Legendre bundle at hand we are able to construct the tangent valued Liouville form

$$\Theta = -p_\alpha^i \mathrm{d}y^\alpha \wedge \omega \otimes \partial_i \tag{5}$$

which can be contracted by $\lambda = \mathrm{d}x^i \otimes (\partial_i + y_i^\alpha \partial_\alpha)$ and one obtains

$$\lambda \rfloor \Theta = -y_i^\alpha p_\alpha^i \omega + p_\alpha^i \mathrm{d}y^\alpha \wedge \omega_i\,. \tag{6}$$

Based on (6) the Hamiltonian form associated to a Lagrangian can be constructed and reads as

$$H_L = \lambda \rfloor \Theta + \mathscr{L}\omega = p_\alpha^i \mathrm{d}y^\alpha \wedge \omega_i - (y_i^\alpha p_\alpha^i - \mathscr{L})\omega = p_\alpha^i \mathrm{d}y^\alpha \wedge \omega_i - \mathscr{H}_L\omega \tag{7}$$

which corresponds to the equation (4) by the Legendre map $p_\alpha^i = \partial_\alpha^i \mathscr{L}$.

The next step is to introduce the polysymplectic form Ω which is defined such that $\Omega\rfloor\phi = -\mathrm{d}(\Theta\rfloor\phi)$ is met with $\phi \in \mathscr{T}^*(\mathscr{X})$. We obtain

$$\Omega\rfloor\phi = \phi_i \mathrm{d}p_\alpha^i \wedge \mathrm{d}y^\alpha \wedge \omega\ ,\ \ \Omega = \mathrm{d}p_\alpha^i \wedge \mathrm{d}y^\alpha \wedge \omega \otimes \partial_i\,. \tag{8}$$

The map H_L yields a fibred morphism $H_L : \mathscr{J}^1(\mathscr{E}) \to Z_\mathscr{E}$

$$Z_\mathscr{E} = \overset{n-1}{\wedge}\mathscr{T}^*(\mathscr{X}) \otimes \mathscr{T}^*(\mathscr{E}) \tag{9}$$

which is termed homogeneous Legendre bundle and possesses the coordinates $(x^i, y^\alpha, p_\alpha^i, p)$ with the additional transition function

$$\bar{p} = \det(\partial_{\bar{i}}\hat{\phi}^i)(p - \partial_i\varphi^{\bar{\alpha}}\partial_{\bar{\alpha}}\hat{\varphi}^\alpha p_\alpha^i) \tag{10}$$

as well as the canonical form

$$\Xi = p\omega + p_\alpha^i dy^\alpha \wedge \omega_i\,. \tag{11}$$

This unique form can be characterized by the fact the $\Xi\rfloor v\rfloor w = 0,\ v, w \in \mathscr{V}(\mathscr{E})$. The horizontal projection h_0 : $\mathrm{d}y^\alpha \to y_i^\alpha \mathrm{d}x^i$ leads to

$$h_0(\Xi) = \left(p + p_\alpha^i y_i^\alpha\right)\omega \tag{12}$$

and this expression shows that all affine maps $\mathscr{J}^1(\mathscr{E}) \to \overset{n}{\wedge}\mathscr{T}^*(\mathscr{X})$ can be expressed in coordinates by (p, p_α^i). Let us choose a section of the bundle $Z_\mathscr{E} \to \mathscr{E}$ with $p = -\mathscr{H}$ and this construction leads to the Hamiltonian form

$$H = p_\alpha^i \mathrm{d}y^\alpha \wedge \omega_i - \mathscr{H}\omega\,. \tag{13}$$

and it is readily observed that in the case of a regular Legendre map (7) and (13) can be related by a given Lagrangian L.

3.1 The differential equations

To obtain the partial differential equations we consider the relation

$$\gamma \rfloor \Omega = \mathrm{d}H\ ,\ \ \gamma = \mathrm{d}x^i \otimes \left(\partial_i + y_i^\alpha \partial_\alpha + p_{\alpha i}^j \partial_j^\alpha\right) \tag{14}$$

where the first jet manifold of the Legendre bundle $\mathscr{J}^1(\Pi)$ possesses the coordinates $(x^i, y^\alpha, p_\alpha^i, y_j^\alpha, p_{\alpha j}^i)$ and from

$$\gamma \rfloor \Omega = \left(\mathrm{d}p_\alpha^l \wedge \mathrm{d}y^\alpha \wedge \omega_l - \mathrm{d}p_\alpha^l y_l^\alpha \wedge \omega + p_{\alpha l}^l \mathrm{d}y^\alpha \wedge \omega\right) \tag{15}$$

we end up with

$$\left((y_i^\alpha - \partial_i^\alpha \mathscr{H})\mathrm{d}p_\alpha^i - \left(p_{\alpha l}^l + \partial_\alpha H\right)\mathrm{d}y^\alpha\right) \wedge \omega = 0 \tag{16}$$

and the partial differential equations follow to

$$y_i^\alpha = \partial_i^\alpha \mathscr{H}\ ,\ \ p_{\alpha i}^i = -\partial_\alpha \mathscr{H}\ . \tag{17}$$

Let us denote by $\hat{\pi}$ the inverse of the map $p_\alpha^i = \partial_\alpha^i \mathscr{L}$, i.e. $y_i^\alpha = \hat{\pi}_i^\alpha(x^i, y^\alpha, p_\alpha^i)$ then from

$$\partial_j^\alpha(\hat{\pi}_i^\beta p_\beta^i - \mathscr{L} \circ \hat{\pi}) = (\partial_j^\alpha(\hat{\pi}_i^\beta) p_\beta^i + \hat{\pi}_j^\alpha - \partial_\rho^i \mathscr{L} \partial_j^\alpha \hat{\pi}_i^\rho) = \hat{\pi}_j^\alpha$$

and

$$\partial_\alpha(\hat{\pi}_i^\beta p_\beta^i - \mathscr{L} \circ \hat{\pi}) = (p_\beta^i \partial_\alpha \hat{\pi}_i^\beta - \partial_\alpha \mathscr{L} - \partial_\rho^i \mathscr{L} \partial_\alpha \hat{\pi}_i^\rho) = -\partial_\alpha \mathscr{L}$$

together with

$$\left(\partial_\alpha - d_i \partial_\alpha^i\right)\mathscr{L} = \left(\partial_\alpha \mathscr{L} - d_i p_\alpha^i\right) = 0$$

it is shown that the equations (3) and (17) correspond in the case of a regular Legendre transformation.

Let us choose a nontrivial connection on $\mathscr{E}$ such that

$$\Gamma = \mathrm{d}x^i \otimes (\partial_i + \Gamma_i^\alpha \partial_\alpha) \tag{18}$$

is met. Therefore one has the splitting

$$H = p_\alpha^i(\mathrm{d}y^\alpha - \Gamma_j^\alpha \mathrm{d}x^j) \wedge \omega_i - \mathscr{H}\omega = p_\alpha^i \mathrm{d}y^\alpha \wedge \omega_i - (p_\alpha^i \Gamma_i^\alpha + \mathscr{H})\omega$$

and the differential equations follow as

$$y_i^\alpha - \Gamma_i^\alpha = \partial_i^\alpha \mathscr{H}\ ,\ \ p_{\alpha i}^i + p_\beta^i \partial_\alpha \Gamma_i^\beta = -\partial_\alpha \mathscr{H}\ . \tag{19}$$

3.2 The conserved quantities

Let us consider the projectable vector field $w = w^i \partial_i + w^\alpha \partial_\alpha$ where we denote its first jet-prolongation with $j^1(w)$. The Lie derivative of the Lagrangian evaluated on solutions of (3) yields

$$(j^1(w))(L) = \left(d_i\left(\partial_\alpha^i \mathscr{L}(w^j y_j^\alpha - w^\alpha) - \mathscr{L} w^i\right)\right)\omega \tag{20}$$

see also [1], which can be written also as

$$(j^1(w))(L) = \left(d_i\left(p_\alpha^i(w^k \partial_k^\alpha \mathscr{H} - w^\alpha) - (p_\alpha^k \partial_k^\alpha \mathscr{H} - \mathscr{H})w^i\right)\right)\omega\ . \tag{21}$$

In the case of mechanics we split the coordinates x^m, $m = 0, \ldots, n$ into t^0 and X^j with the convention that the indices $j = 1 \ldots n$ in contrast to the general case where the indices meet $j = 0, \ldots, n$, since we explicitly label the coordinate $x^0 = t^0$. If the Lagrangian is independent of the time, i.e $\partial_0 L = 0$, where we assume a trivial connection Γ, then we obtain with $w^0 = 1$, $w^j = 0$ and $w^\alpha = 0$ the relation

$$0 = \partial_0 L = d_0\left(p_\alpha^0 \partial_0^\alpha \mathscr{H} - (p_\alpha^j \partial_j^\alpha \mathscr{H} + p_\alpha^0 \partial_0^\alpha \mathscr{H} - \mathscr{H})\right)$$

and consequently

$$d_0\left(-p_\alpha^j \partial_j^\alpha \mathscr{H} + \mathscr{H}\right) = 0$$

shows that the expression

$$\mathscr{H} - p_\alpha^j \partial_j^\alpha \mathscr{H} = p_\alpha^0 y_0^\alpha - \mathscr{L} \tag{22}$$

is the conserved quantity for this special case. We have shown that in the case where the Lagrangian is time-independent the conserved quantity reads as

$$\mathfrak{H} = p_\alpha^0 y_0^\alpha - \mathscr{L}$$

which corresponds to the total energy in many applications. It is worth mentioning that the Hamiltonian $\mathscr{H}$ and the conserved $\mathfrak{H}$ quantity differ in the expressions $p_\alpha^i y_i^\alpha$ containing the spatial momenta Thus we are motivated to construct a Hamiltonian formulation where the Hamiltonian equals the conserved quantity in the time invariant case. This will be demonstrated in the following section.

4 Hamiltonian Evolution Equations

A different view of Hamiltonian field theory is obtained when the equations are formulated with respect to the evolution of time only. Let us denote the special Hamiltonian which contains only momenta with respect to the time coordinate as

$$\mathfrak{H} = p_\alpha^0 y_0^\alpha - \mathscr{L} \tag{23}$$

then it is verified that the evolution equations can be written as

$$y_0^\alpha = \partial_0^\alpha \mathfrak{H}\,, \;\; p_{\alpha 0}^0 = -\delta_\alpha \mathfrak{H}\,. \tag{24}$$

To show this let us denote by $\hat{\theta}$ the inverse of the map $p_0^\alpha = \partial_0^\alpha \mathscr{L}$, i.e. $y_0^\alpha = \hat{\theta}_0^\alpha(x^i, y^\alpha, p_0^\alpha, y_i^\alpha)$ then from

$$\partial_0^\alpha(\hat{\theta}_0^\beta p_\beta^0 - \mathscr{L}\circ\hat{\theta}) = (\partial_0^\alpha(\hat{\theta}_0^\beta)p_\beta^0 + \hat{\theta}_0^\alpha - \partial_\rho^0\mathscr{L}\partial_0^\alpha\hat{\theta}_i^\rho) = \hat{\theta}_0^\alpha$$

and

$$\partial_\alpha(\hat{\theta}_0^\beta p_\beta^0 - \mathscr{L}\circ\hat{\theta}) = (p_\beta^0\partial_\alpha\hat{\theta}_0^\beta - \partial_\alpha\mathscr{L} - \partial_\rho^0\mathscr{L}\partial_\alpha\hat{\theta}_0^\rho) = -\partial_\alpha\mathscr{L}$$

as well as from

$$d_i\partial_\alpha^i(\hat{\theta}_0^\beta p_\beta^0 - \mathscr{L}\circ\hat{\theta}) = d_i(p_\beta^0\partial_\alpha^i\hat{\theta}_0^\beta - \partial_\alpha^i\mathscr{L} - \partial_\rho^0\mathscr{L}\partial_\alpha^i\hat{\theta}_0^\rho) = -d_i\partial_\alpha^i\mathscr{L}$$

the equivalence is shown.

4.1 The geometric background

To explain the geometric motivation behind this approach let us consider a different bundle structure and we consider $\mathscr{U} \to \mathscr{D}$ where in contrast to the last section the manifold $\mathscr{D}$ with $\dim(\mathscr{D}) = s$ only consists of the spatial variables which are denoted as X^i with $i = 1, \ldots, s$ and the manifold $\mathscr{U}$ is equipped with the coordinates u^α. The volume form is now denoted by

$$\Omega = \mathrm{d}X^1 \wedge \ldots \wedge \mathrm{d}X^n\,, \;\; \Omega_i = \partial_i \rfloor \Omega$$

and in the Lagrangian picture we consider the bundle structure $\mathscr{U} \to \mathscr{D}$ with $\left(X^i, y^\alpha, \dot{y}^\alpha\right) \to X^i$ where it is worth mentioning that we have the identification $\dot{y}^\alpha = y_0^\alpha$ and $p_\alpha = p_\alpha^0$ when the flow parameter of the semi-group corresponds to the time t^0, but the bundle construction remains different of course. Let us consider a section $\sigma : \mathscr{D} \to \mathscr{U}$ together with the total time change of the Hamiltonian functional, which is given as

$$\int_{\mathscr{D}} \left(j^2\sigma\right)^* \left(j^1(v)(\mathfrak{H}\Omega)\right) = \int_{\mathscr{D}} \left(j^2\sigma\right)^* \left(j^1(v)\rfloor d(\mathfrak{H}\Omega)\right)\,, \tag{25}$$

for first order Hamiltonians, and we consider an evolutionary vertical vector field $v : \mathscr{U} \to \mathscr{V}(\mathscr{U})$ together with the first prolongation $j^1(v) = v^\alpha\partial_\alpha + d_i(v^\alpha)\partial_\alpha^i$. Let us inspect the expression

$$j^1(v)\rfloor \mathrm{d}(\mathfrak{H}\Omega) = (v^\alpha\partial_\alpha\mathfrak{H} + d_i(v^\alpha)\partial_\alpha^i\mathfrak{H})\Omega$$

and integration by parts leads to

$$j^1(v)\rfloor \mathrm{d}(\mathfrak{H}\Omega) = (v^\alpha\partial_\alpha\mathfrak{H} - v^\alpha d_i\partial_\alpha^i\mathfrak{H})\Omega + d_i(v^\alpha\partial_\alpha^i\mathfrak{H})\Omega\,. \tag{26}$$

Using the variational derivative δ and the horizontal derivative d_h the equation can be rewritten as

$$j^1(v)\rfloor \mathrm{d}(\mathfrak{H}\Omega) = v\rfloor \delta_\alpha\mathfrak{H}\mathrm{d}u^\alpha \wedge \Omega + \mathrm{d}_h(v\rfloor \partial_\alpha^i\mathfrak{H}\mathrm{d}u^\alpha \wedge \Omega_i)\,,$$

where we have the coordinate expression

$$\mathfrak{H}\Omega \to \delta_\alpha\mathfrak{H}\mathrm{d}u^\alpha \wedge \Omega\,, \quad \delta_\alpha = \partial_\alpha - d_i(\partial_\alpha^i)\,.$$

It is easily seen that the total derivative d splits into the variational derivative δ and an exact form. Furthermore the additional map δ^∂ can be introduced with

$$\mathfrak{H}\Omega \to \delta^\partial(\mathfrak{H}\Omega)\,, \quad \delta^\partial(\mathfrak{H}\Omega) = \partial_\alpha^i\mathfrak{H}\mathrm{d}u^\alpha \wedge \Omega_i\,.$$

Therefore we can conclude that in first order mechanics we obtain

$$\delta^\partial(\mathfrak{H}\Omega) = \partial_\alpha^i\mathfrak{H}\mathrm{d}y^\alpha \wedge \Omega_i = -p_\alpha^i\mathrm{d}y^\alpha \wedge \Omega_i$$

which shows that the spatial momenta also appear in the boundary map.

5 Continuum Mechanics

In this part we want to discuss the Lagrangian description of continuum mechanics in an intrinsic form, see also [8], by considering the formulations presented so far in this paper. We start with some geometric preliminaries necessary for an intrinsic description of mechanics.

5.1 The geometry of Continuum Mechanics

The configuration bundle $\mathscr{Q} \to \mathscr{B}$ possesses the coordinates $(t^0, q^\alpha) \to (t^0)$ and the reference bundle $\mathscr{R} \to \mathscr{B}$ is introduced with coordinates (t^0, X^i) for $\mathscr{R}$. We construct the bundle $\mathscr{C}_e \to \mathscr{R}$, with

$$\mathscr{C}_e = \mathscr{Q} \times_{\mathscr{B}} \mathscr{R} \tag{27}$$

and coordinates (t^0, X^i, q^α) for $\mathscr{C}_e$. Furthermore, we introduce the following geometric objects. A symmetric vertical metric g on the fibres of $\mathscr{Q} \to \mathscr{B}$, a trivial reference frame $\gamma = \mathrm{d}t^0 \otimes \partial_0$ and a connection splitting the vertical tangent bundle $\mathscr{V}(\mathscr{Q}) \to \mathscr{Q}$, which is denoted by Λ. In coordinates we obtain for the metric

$$g = g_{\alpha\beta} \mathrm{d}q^\alpha \otimes \mathrm{d}q^\beta \ , \ \ g_{\alpha\beta} \in \mathscr{C}^\infty(\mathscr{Q})$$

and for the connection

$$\Lambda = \mathrm{d}t^0 \otimes \partial_0 + \mathrm{d}q^\alpha \otimes (\partial_\alpha + \Lambda_\alpha^\rho \dot{\partial}_\rho) \ , \ \ \Lambda_\alpha^\rho \in \mathscr{C}^\infty(\mathscr{Q})$$

with $\partial_0 g_{\alpha\beta} = 0$ and the volume form reads as

$$\mathrm{vol} = \sqrt{\left|\det(g_{\alpha\beta})\right|} \mathrm{d}q^1 \wedge \ldots \wedge \mathrm{d}q^n \ , \ \ g_{\alpha\beta} \in \mathscr{C}^\infty(\mathscr{Q}) \, .$$

On the reference bundle we have a metric on the fibres of $\mathscr{R}$

$$G = G_{ij} \mathrm{d}X^i \otimes \mathrm{d}X^j \ , \ \ G_{ij} \in \mathscr{C}^\infty(\mathscr{R})$$

as well as a volume form

$$\mathrm{VOL} = \sqrt{\left|\det(G_{ij})\right|} \Omega \ , \ \ G_{ij} \in \mathscr{C}^\infty(\mathscr{R}) \, .$$

For Λ we choose a linear connection

$$2\Lambda_{\alpha\rho}^\kappa = -g^{\kappa\beta}\left(\partial_\alpha g_{\rho\beta} + \partial_\rho g_{\beta\alpha} - \partial_\beta g_{\alpha\rho}\right)$$

and for simplicity we only discuss the case $\dim(\mathscr{Q}) = \dim(\mathscr{R})$.

A motion in the Lagrangian setting is a map $\Phi : \mathscr{R} \to \mathscr{C}_e$ with

$$q^\alpha = \Phi^\alpha\left(t^0, X^i\right)$$

and the tangent map of $\Phi : \mathscr{R} \to \mathscr{C}_e$ is given as $T(\Phi) : \mathscr{T}(\mathscr{R}) \to \mathscr{T}(\mathscr{C}_e)$

$$T(\Phi) = \mathrm{d}t^0 \otimes (\partial_0 + V_0^\alpha \partial_\alpha) + \mathrm{d}X^i \otimes (\partial_i + F_i^\alpha \partial_\alpha) \, .$$

5.1.1 Stress forms

Let us consider the Cauchy stress form

$$\sigma = \sigma^{\alpha\beta} \partial_\alpha \rfloor \mathrm{vol} \otimes \partial_\beta \tag{28}$$

together with the map $\Phi : \mathscr{R} \to \mathscr{C}_e$ that allows to pull back the form part of (28). This leads to the 1st Piola stress tensor

$$P = \Phi^*\left(\sigma^{\alpha\beta} \partial_\alpha \rfloor \mathrm{vol}\right) \otimes \partial_\beta = P^{i\beta} \partial_i \rfloor \mathrm{VOL} \otimes \partial_\beta \, . \tag{29}$$

The 2nd Piola stress tensor is given as

$$S = \Phi^*\left(\sigma^{\alpha\beta} \partial_\alpha \rfloor \mathrm{vol} \otimes \partial_\beta\right) = S^{ij} \partial_i \rfloor \mathrm{VOL} \otimes \partial_j \tag{30}$$

and the relation $S^{ij} = P^{i\beta} \hat{F}_\beta^j$ is met. The Cauchy Green tensor is obtained by pulling back the metric g by the map $\Phi : \mathscr{R} \to \mathscr{C}_e$. $C_{ij} = (g_{\alpha\beta} \circ \Phi) F_i^\alpha F_j^\beta$. Therefore the following quantities are adopted which do not require the knowledge of Φ. We have

$$\check{P}^{i\beta} \circ j^1\Phi = P^{i\beta} \ , \ \ \check{S}^{ij} \circ j^1\Phi = S^{ij} \ , \ \ \check{C}_{ij} \circ j^1\Phi = C_{ij} \tag{31}$$

which means that $\check{P}^{i\beta}, \check{S}^{ij}, \check{C}_{ij} \in C^\infty(\mathscr{J}^1(\mathscr{C}_e))$. Then the relation $P^{i\beta} = S^{ij} F_j^\beta$ reads $\check{P}^{i\beta} = \check{S}^{ij} q_j^\beta$.

5.2 The Euler Lagrange Equations

We investigate only first order Lagrangians and the variational derivative for this setting looks in coordinates as

$$\delta_\alpha = \partial_\alpha - d_i \partial_\alpha^i - d_0 \partial_\alpha^0$$

with

$$d_i = \partial_i + q_i^\alpha \partial_\alpha \,, \;\; d_0 = \partial_0 + q_0^\alpha \partial_\alpha \,.$$

Let us consider the density of the kinetic energy

$$E_{k_d} = \rho_{\mathscr{R}} E_k \mathrm{VOL} \wedge \mathrm{d}t^0 \,, \;\; E_k = \frac{1}{2} q_0^\alpha g_{\alpha\beta} q_0^\beta \tag{32}$$

with the mass density $\rho_{\mathscr{R}}$, together with the balance of mass $\partial_0 \rho_{\mathscr{R}} = 0$, and the stored energy function which meets

$$2\rho_{\mathscr{R}} \frac{\partial}{\partial \check{C}_{ij}} E_{el} = \check{S}^{ij} \,. \tag{33}$$

We use the variational principle

$$\delta_\alpha(\mathscr{L}) + \rho_{\mathscr{R}} \sqrt{|\det(G_{ij})|} g_{\alpha\rho} B^\rho = 0 \,,$$

where the body force densitiy has been added, with

$$\mathscr{L} = \rho_{\mathscr{R}} \sqrt{|\det(G_{ij})|} \, (E_k - E_{el}) \tag{34}$$

which consequently leads to

$$\rho_{\mathscr{R}} \left(q_{00}^\eta - \Lambda_{\beta\sigma}^\eta q_0^\beta q_0^\sigma \right) = d_k \check{P}^{k\eta} + \check{P}^{k\eta} \Lambda_{k\beta}^\beta - \check{P}^{k\tau} \Lambda_{\tau\beta}^\eta q_k^\beta + \rho_{\mathscr{R}} B^\eta \,. \tag{35}$$

Using the spatial picture we obtain

$$\rho_{\mathscr{R}} \left(\partial_0 V_0^\rho - V_0^\alpha V_0^\beta \Lambda_{\alpha\beta}^\rho \right) = B^\rho \rho_{\mathscr{R}} + \partial_i P^{i\rho} - P^{i\rho} \Lambda_{ir}^r - P^{i\tau} \Lambda_{\kappa\tau}^\rho F_i^\kappa \,, \tag{36}$$

which are partial differential equations in the unknown functions Φ^α and we have $V_0^\alpha = \partial_0 \Phi^\alpha$ and $F_i^\alpha = \partial_i \Phi^\alpha$.

5.3 The polysymplectic point of View

Again we have coordinates for $\mathscr{X}$ which read as $(x^0 = t^0, x^i = X^i)$ and we choose the trivial connection

$$\Gamma = \mathrm{d}t^0 \otimes \partial_0 + \mathrm{d}X^i \otimes \partial_i \,.$$

From the Lagrangian (34) we derive the spatial momenta

$$p_\alpha^i = -\partial_\alpha^i \left(\rho_{\mathscr{R}} E_{el} \sqrt{|\det(G_{ij})|} \right) = -(\sqrt{|\det(G_{ij})|} \check{S}^{ij} g_{\alpha\tau} q_j^\tau) \,.$$

as well as the temporal momenta

$$p_\alpha^0 = \partial_\alpha^0 \left(\rho_{\mathscr{R}} E_k \sqrt{|\det(G_{ij})|} \right) = \rho_{\mathscr{R}} \sqrt{|\det(G_{ij})|} g_{\alpha\beta} q_0^\beta$$

and the Hamiltonian can be computed in the case where the second Piola stress tensor is invertible from the expression

$$\mathscr{H} = p_\alpha^0 q_0^\alpha + p_\alpha^i q_i^\alpha - (\rho_{\mathscr{R}} \sqrt{|\det(G_{ij})|} \left(\frac{1}{2} q_0^\alpha g_{\alpha\beta} q_0^\beta - E_{el} \right)$$

which gives

$$\mathscr{H} = \frac{1}{\sqrt{|\det(G_{ij})|}} \left(\frac{1}{2\rho_R} p_\alpha^0 p_\beta^0 g^{\alpha\beta} - p_\alpha^i p_\rho^j g^{\rho\alpha} \check{S}_{ij} \right) + \rho_{\mathscr{R}} \sqrt{|\det(G_{ij})|} E_{el} \,.$$

It is important to remark that the case of a degenerate Lagrangian can be handled using advanced tools, which can be found for example in [4] where it is shown that the problem of regularity can be treated using a more general equivalence of Lagrangian and Hamiltonian Systems not necessary having the same order of the Lagrangians and the associated Hamiltonians. A different approach where the authors introduce a Lagrangian constrained space which is the image of the Legendre map and clearly a subset of Π, can be found in [1]. This problem of regularity will not be discussed he here and we proceed with the assumption of a non degenerate Lagrangian such that the equations read as

$$q_0^\alpha = \partial_0^\alpha \mathscr{H} \quad , \quad q_i^\alpha = \partial_i^\alpha \mathscr{H} \quad , \quad p_{\alpha i}^i + p_{\alpha 0}^0 = -\partial_\alpha \mathscr{H} + \sqrt{\left|\det(G_{ij})\right|}\rho_{\mathscr{R}} g_{\alpha\eta} B^\eta \,. \tag{37}$$

To show the last equation of (37) we collect again the terms and have

$$p_{\alpha i}^i = d_i(\sqrt{\left|\det(G_{ij})\right|}\check{P}^{i\tau})g_{\alpha\tau} + \sqrt{\left|\det(G_{ij})\right|}\check{P}^{i\tau}\partial_\beta(g_{\alpha\tau})q_i^\beta \;.$$

$$p_{\alpha 0}^0 = \rho_{\mathscr{R}}\sqrt{\left|\det(G_{ij})\right|}d_0\left(g_{\alpha\beta}q_0^\beta\right)$$

$$\partial_\alpha \mathscr{H} = \frac{1}{2}\frac{1}{\sqrt{\left|\det(G_{ij})\right|}}\left(\frac{1}{\rho_{\mathscr{R}}}p_\alpha^0 p_\beta^0 \partial_\alpha g^{\alpha\beta} + \check{S}_{lm} p_\rho^l p_\tau^m \partial_\alpha g^{\rho\tau}\right)$$

which produces the same equation as the relation (35).

5.4 The evolutionary point of view

Let us start to introduce the momentum in the spatial description which can be given as

$$p_\alpha = g_{\alpha\beta} q_0^\beta \rho_{\mathscr{R}} \sqrt{\left|\det(G_{ij})\right|}\,. \tag{38}$$

In the material description we obtain

$$P_\alpha = p_\alpha \circ j^1(\Phi) = \left(g_{\alpha\beta}\circ\Phi\right) V_0^\beta \rho_{\mathscr{R}} \sqrt{\left|\det(G_{ij})\right|}\;,$$

where the symbol of the momentum P should not be confused with the one of the Piola tensor. The total energy is the sum of the kinetic and the stored energy function and consequently the Hamiltonian and reads as

$$\mathfrak{H} = \frac{1}{2}\frac{g^{\rho\kappa} p_\kappa p_\rho}{\rho_{\mathscr{R}}\sqrt{\left|\det(G_{ij})\right|}} + E_{el}\rho_{\mathscr{R}}\sqrt{\left|\det(G_{ij})\right|}\,.$$

The equations of motion can be written as

$$\partial_0 \Phi^\beta = \left(\delta^\beta \mathfrak{H}\right)\circ j^1(\Phi)$$

and

$$\partial_0 P_\beta = -\left(\delta_\beta \mathfrak{H}\right)\circ j^1(\Phi) + \sqrt{\left|\det(G_{ij})\right|}\rho_{\mathscr{R}} g_{\beta\eta} B^\eta$$

which are the counterpart to the relations (36). In the set of equations the variational derivatives read as

$$\delta^\beta = \dot{\partial}^\beta = \partial_0^\beta \;,\;\; \delta_\beta = \partial_\beta - d_i \partial_\beta^i \,.$$

It is worth mentioning that in the case where the Legendre map only consists of the temporal momenta the problem of regularity is much easier since it only depends on the mass metric tensor.

6 Conclusion

In this contribution we presented an analysis of different concepts regarding first order field theory in mechanics. Further directions of research will include beside an analysis of higher order problems preeminently a detailed discussion of the boundary conditions. Also the problem of regularity should be analyzed and it will be interesting to apply the proposed methods [1, 4] to continuum mechanics.

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