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PERIODICAL OSCILLATIONS OF CONTROL SYSTEMS. ANALYTICAL AND NUMERICAL APPROACH.

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Abstract. The method of harmonic linearization, Lyapunov quantities, numerical methods, and the applied bifurcation theory together discover new opportunities for analysis of periodic oscillations of control systems. In the present work these opportunities are demonstrated.

Here the quadratic system is reduced to the Lienard equation and by the latter the two-dimensional domain of parameters, corresponding the existence of four limit cycles: three "small" and one "large", was evaluated.

This criterion together with numerical estimates of oscillations amplitude give an estimate of occurrence of chaotic oscillations in the Henon system. In the work it is also considered the Feigenbaum effect for nonunimodal maps which describe discrete phase-locked loops.

1 Introduction

At present, the analytical methods for analysis of stability of automatic control systems are sufficiently well developed. These are frequency criteria of absolute stability [6, 14, 16]. But quite another situation holds in the case of analysis of periodic and chaotic oscillations of control systems. The method of harmonic linearization (describing functions method), being also frequency in form, has considerable restrictions and the results beyond these restrictions can be wrong. The method of harmonic linearization, numerical methods, and the applied bifurcation theory together [11, 12] discover new opportunities for analysis of periodic oscillations of control systems. In the present report these opportunities are demonstrated. We also formulate the circle criterion of nonexistence of periodic oscillations of fixed period in discrete control systems. This criterion together with numerical estimates of oscillations amplitude gives an estimate of occurrence of chaotic oscillations in the Henon system. In the report it is also considered the Feigenbaum effect for nonunimodal maps which describe discrete phase locked loops.

Here the quadratic system is reduced to the Lienard equation and by the latter the two-dimensional domain of parameters, corresponding the existence of four limit cycles: three "small" and one "large", was evaluated. This domain extends the domain of parameters obtained for the quadratic system with four limit cycles due to Shi in 1980.

2 Large and small limit cycles in quadratic system

The study of limit cycles of two-dimensional dynamical systems was stimulated by purely mathematical problems (the center-and-focus problem, Hilbert's sixteenth problem, and isochronous centers problem) as well as many applied problems (the oscillations of electronic generators and electrical machines, the dynamics of populations).

One of the central problems [4, 5, 8, 18, 19, 20, 24] in studying small cycles in the neighborhood of equilibrium is a computation of the Lyapunov quantities (or Poincare-Lyapunov constants).

In the present work the method of Lyapunov quantities is applied to investigation of small and large limit cycles.

Consider a system of two autonomous differential equations

$$\begin{aligned} \frac{dx}{dt} &= -y + f(x,y), \\ \frac{dy}{dt} &= x + g(x,y), \end{aligned} \tag{1}$$

where $x, y \in \mathbb{R}$ and the functions $f(\cdot,\cdot)$ and $g(\cdot,\cdot)$ are sufficiently smooth.

Suppose, the expansion of the functions f, g begins with the terms of at least the second order and therefore we have

$$\begin{aligned} & f(0,0) = g(0,0) = 0, \\ & \frac{df}{dx}(0,0) = \frac{df}{dy}(0,0) = \frac{dg}{dx}(0,0) = \frac{dg}{dy}(0,0) = 0. \end{aligned} \tag{2}$$

and

$$\begin{aligned} f(x,y) &= \sum_{k+j=2}^{n} f_{kj}x^k y^j + o\big((|x|+|y|)^n\big) \\ g(x,y) &= \sum_{k+j=2}^{n} g_{kj}x^k y^j + o\big((|x|+|y|)^n\big) \end{aligned} \tag{3}$$

While the first and second Lyapunov quantities had been computed in the 40-50s of last century [2, 22], the third Lyapunov quantity was computed in terms of f_{ij} and g_{ij} in [7, 15] and its expression occupies more than four pages and the expression for the fourth Lyapunov quantity occupies 45 pages.

Here we will consider the expressions of Lyapunov quantities $\mathrm{L}_{i=1,\dots,5}$ for Lienard system, which are calculated by symbolic calculation packages.

Assuming in (1)

$$f(x,y) \equiv 0$$

$$\frac{dg(x,y)}{dy} = g_{x1}(x),\ g(x,0) = g_{x0}(x),\ \frac{dg_{x0}}{dx}(0) = 0,$$

we obtain the following system

$$\begin{aligned} &\dot{x} = -y, \\ &\dot{y} = x + g_{x1}(x)y + g_{x0}(x), \end{aligned} \tag{4}$$

Let $g_{x1}(x) = g_{11}x + \dots,\ g_{x0}(x) = g_{11}x^2 + \dots$ Then

$$\mathrm{L}_1 = -\frac{\pi}{4}(g_{20}g_{11} - g_{21}).$$

If $g_{21} = g_{20}g_{11}$, then $\mathrm{L}_1 = 0$ and

$$\mathrm{L}_2 = \frac{\pi}{24}(3g_{41} - 5g_{20}g_{31} - 3g_{40}g_{11} + 5g_{20}g_{30}g_{11}).$$

If $g_{41} = \frac{5}{3}g_{20}g_{31} + g_{40}g_{11} - \frac{5}{3}g_{20}g_{30}g_{11}$, then $\mathrm{L}_2 = 0$ and

$\mathrm{L}_3 = -\frac{\pi}{576}(70g_{20}^3g_{30}g_{11} + 105g_{20}g_{51} + 105g_{30}^2g_{11}g_{20} + 63g_{40}g_{31} - 63g_{11}g_{40}g_{30} - 105g_{30}g_{31}g_{20} - 70g_{20}^3g_{31} - 45g_{61} - 105g_{50}g_{11}g_{20} + 45g_{60}g_{11})$.

If g_{61} is determined from equation $\mathrm{L}_3 = 0$ then

$\mathrm{L}_4 = \frac{\pi}{17280}(945g_{81} + 4158g_{20}^2g_{40}g_{31} + 2835g_{20}g_{30}g_{51} - 5670g_{20}g_{30}g_{11}g_{50} - 4158g_{20}^2g_{30}g_{11}g_{40} + 2835g_{20}g_{11}g_{70} + 1215g_{30}g_{11}g_{60} + 1701g_{40}g_{11}g_{50} - 4620g_{20}^3g_{11}g_{50} - 8820g_{20}^3g_{30}g_{31} + 1701g_{30}g_{40}g31 + 2835g_{20}g_{50}g_{31} - 2835g_{20}g_{30}^2g_{31} - 1701g_{30}^2g_{11}g_{40} + 8820g_{20}^3g_{30}^2g_{11} + 3080g_{20}^5g_{11}g_{30} + 2835g_{20}g_{30}^3g_{11} + 4620g_{20}^3g_{51} - 1701g_{40}g_{51} - 945g_{11}g_{80} - 3080g_{20}^5g_{31} - 1215g_{60}g_{31} - 2835g_{20}g_{71})$.

If g_{81} is determined from equation $\mathrm{L}_4 = 0$ then

$\mathrm{L}_5 = \frac{\pi}{3110400}(-1621620g_{20}^2g_{40}g_{11}g_{50} - 3118500g_{20}^2g_{30}g_{40}g_{31} - 935550g_{20}g_{30}g_{11}g_{70} + 2522520g_{20}^4g_{30}g_{11}g_{40} - 935550g_{20}g_{30}g_{50}g_{31} - 486486g_{20}g_{30}g_{40}^2g_{11} + 1403325g_{20}g_{30}^2g_{11}g_{50} - 579150g_{20}^2g_{30}g_{11}g_{60} + 5128200g_{20}^3g_{30}g_{11}g_{50} - 561330g_{30}g_{40}g_{11}g_{50} + 127575g_{101} + 1351350g_{20}^3g_{71} - 127575g_{11}g_{100} - 280665g_{40}g_{71} - 200475g_{60}g_{51} - 2402400g_{20}^5g_{51} + 1601600g_{20}^7g_{31} - 155925g_{80}g_{31} - 1601600g_{20}^7g_{11}g_{30} + 2402400g_{20}^5g_{11}g_{50} - 467775g_{20}g_{50}^2g_{11} - 4158000g_{20}^3g_{30}^3g_{11} + 1621620g_{20}^2g_{40}g_{51} + 467775g_{20}g_{70}g_{31} - 467775g_{20}g_{30}^2g_{51} + 467775g_{20}g_{30}^3g_{31} + 200475g_{30}g_{60}g_{31} + 5613300g_{20}^5g_{30}g_{31} + 467775g_{20}g_{30}g_{71} - 2113650g_{20}^3g_{50}g_{31} + 280665g_{40}g_{50}g_{31} - 2522520g_{20}^4g_{40}g_{31} - 280665g_{30}^2g_{40}g_{31} + 486486g_{20}g_{40}^2g_{31} - 5613300g_{20}^5g_{30}^2g_{11} + 467775g_{20}g_{11}g_{90} + 280665g_{30}^3g_{11}g_{40} - 467775g_{20}g_{30}^4g_{11} + 467775g_{20}g_{50}g_{51} + 280665g_{30}g_{40}g_{51} - 3014550g_{20}^3g_{30}g_{51} + 4158000g_{20}^3g_{30}^2g_{31} + 579150g_{20}^2g_{60}g_{31} - 1351350g_{20}^3g_{11}g_{70} + 280665g_{40}g_{11}g_{70} + 200475g_{50}g_{11}g_{60} - 200475g_{30}^2g_{11}g_{60} + 155925g_{30}g_{11}g_{80} + 3118500g_{20}^2g_{30}^2g_{11}g_{40} - 467775g_{20}g_{91})$.

2.1 Transformation between quadratic system and the Lienard system

Let us consider transformation of quadratic system to a special type of Lienard system

$$\begin{aligned} &\dot{x} = y, \\ &\dot{y} = -F(x)y - G(x), \end{aligned} \tag{5}$$

where

$$\begin{aligned} &F(x) = (Ax+B)x|x+1|^{q-2}, \\ &G(x) = (C_1x^3 + C_2x^2 + C_3x + 1)x\frac{|x+1|^{2q}}{(x+1)^3}. \end{aligned} \tag{6}$$

We have the following results [10], [15].

Lemma 1. *Suppose, for the coefficients A, B, C_1, C_2, C_3, q of equation (5) the relations*

$$\frac{(B-A)}{(2q-1)^2}((1-q)B+(3q-2)A) = 2C_2 - 3C_1 - C_3, \tag{7}$$

$$\frac{(B-A)}{(2q-1)^2}(B+2(q-1)A) = C_2 - 2C_1 - 1. \tag{8}$$

are satisfied. Then, by a nonsingular change, equation (5) can be reduced to the quadratic system

$$\begin{aligned} \dot{x} &= p(x,y) = a_1x^2 + b_1xy + \alpha_1 x + \beta_1 y, \\ \dot{y} &= q(x,y) = a_2x^2 + b_2xy + c_2y^2 + \alpha_2 x + \beta_2 y. \end{aligned} \tag{9}$$

with the coefficients $b_1 = 1$, $\alpha_1 = 1$, $\beta_1 = 1$, $c_2 = -q$, $\alpha_2 = -2, \beta_2 = -1$,

$$\begin{aligned} a_1 &= 1 + \frac{B-A}{2q-1}, \\ a_2 &= -(q+1)a_1^2 - Aa_1 - C_1, \\ b_2 &= -A - a_1(2q+1). \end{aligned} \tag{10}$$

Then, by the above relations for the Lyapunov quantities L_1 and L_2, we obtain the following.

Lemma 2. *If $L_1 = L_2 = 0$, $5A - 2Bq - 4B = 0$ and $A \neq B$, $AB \neq 0$, $q \neq \frac{1}{2}$ then*

$$C_1 = (q+3)\frac{B^2}{25} - \frac{(1+3q)}{5},$$

$$C_2 = \left(15(1-2q) + 3B^2\right)\frac{1}{25},$$

$$C_3 = \frac{3(3-q)}{5};$$

$$\mathrm{L}_3 = -\frac{\pi B(q+2)(3q+1)[5(q+1)(2q-1)^2 + B^2(q-3)]}{20000}.$$

Thus, if the conditions of Lemma 2 and $\mathrm{L}_3 \neq 0$, then by small disturbances of system we can obtain three "small" cycles around the zero equilibrium of system and seek "large" cycles on a plane of the rest two coefficients (B,q).

Lemma 3. *For $b_1 \neq 0$ system (9) can be reduced to the Lienard equation (5) with the functions*

$$F(x) = R(x)e^{p(x)} = R(x)|\beta_1 + b_1x|^q,$$

$$G(x) = P(x)e^{2p(x)} = P(x)|\beta_1 + b_1x|^{2q}.$$

Here $q = -\dfrac{c_2}{b_1}$,

$$R(x) = -\frac{(b_1b_2 - 2a_1c_2 + a_1b_1)x^2 + (b_2\beta_1 + b_1\beta_2 - 2\alpha_1c_2 + 2a_1\beta_1)x + \alpha_1\beta_1 + \beta_1\beta_2}{(\beta_1 + b_1x)^2},$$

$$P(x) = -\left(\frac{a_2x^2 + \alpha_2 x}{\beta_1 + b_1x} - \frac{(b_2x + \beta_2)(a_1x^2 + \alpha_1 x)}{(\beta_1 + b_1x)^2} + \frac{c_2(a_1x^2 + \alpha_1 x)^2}{(\beta_1 + b_1x)^3}\right).$$

2.2 Computer experiments

The above results were applied to quadratic systems and the experiments for computing "large" cycles were carried out.

In these experiments the reduction of quadratic system to the Lienard equation of special form (5)-(6) was used and with its help a set of parameters B, q (Figure 1), which correspond to the existence of "large" cycle, was estimated.

In Figure 1 it is shown a domain bounded by straight lines, which correspond to the lines of reversal sign of the third Lyapunov quantity. The curve C in the graph is a curve of the parameters B and q of the Lienard system,

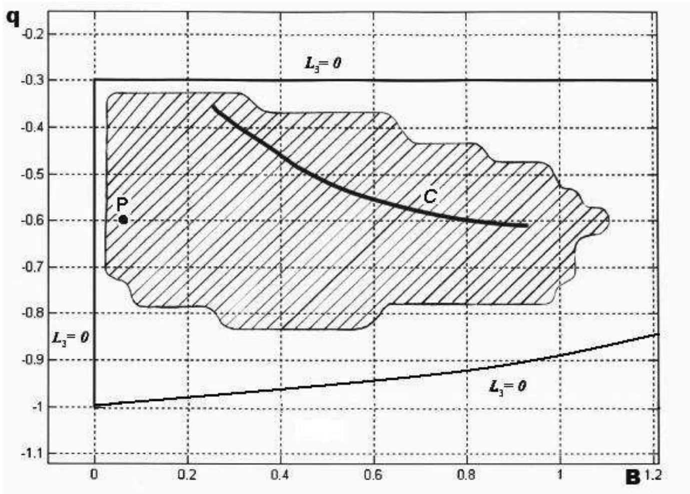

Figure 1: Domain of existence of "large" limit cycles.

which correspond to parameters of quadratic system, such that for these parameters the results of the existence of four cycles were obtained in [21].

Since two Lyapunov quantities are equal to zero, by small disturbances it is possible to construct systems with four cycles for the considered domain of parameters: three small cycle around one equilibrium and one large cycle around another equilibrium.

Note that if the conditions of Lemma 2 are satisfied, then the changes of the time $t \to -t$ and the parameter of system $B \to -B$ don't modify system (6). Therefore, analogous domain of existence of large cycle, which is symmetric about the straight line $B = 0$, holds.

These results were applied to quadratic systems and the experiments for computing "large" cycles were carried out. Our experience of computations shows that it is practically impossible to trace "small" cycles in the neighborhood of equilibrium, where the zero and the first Lyapunov quantities are equal to zero. However in a number of computer experiments we can distinctly see "large" cycles.

For example, in Figure 2 it is shown a "large" cycle for the system

$$\begin{aligned}\dot{x} &= 0.99x^2 + xy + y,\\ \dot{y} &= 0.58x^2 + 0.17xy + 0.6y^2 - 2x - y,\end{aligned}$$

the parameters of which correspond to the point P in Figure 2.

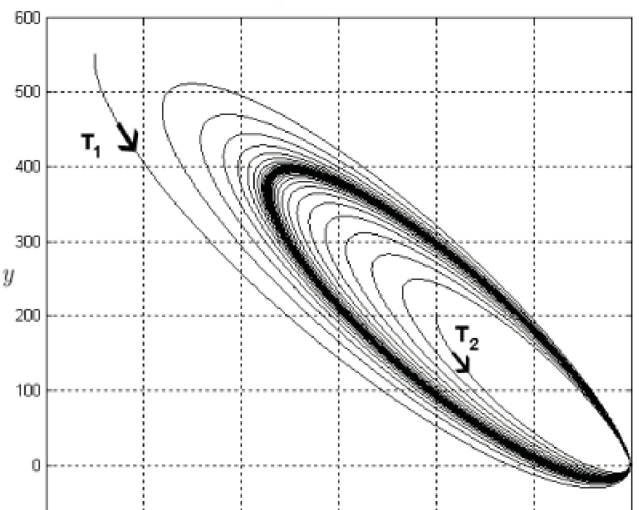

Figure 2: Stable limit cycle in quadratic system.

3 The harmonic linearization method — describing function method

Consider a system

$$\frac{dx}{dt} = Px + q\varphi(r^*x), \tag{1}$$

where P is a constant $n \times n$-matrix, q and r are constant n-vectors, $\varphi(\sigma)$ is a piecewise-continuous function, and $*$ is the operation of transposition.

When applied the harmonic linearization method [6] to this system, standard assumptions are the existence of a pair of purely imaginary eigenvalues $\pm i\omega_0 (\omega_0 > 0)$ of the matrix P and a negativeness of the rest of eigenvalues.

By nonsingular linear transformation, under the above assumptions system (1) can be reduced to the form

$$\begin{aligned} \dot{x}_1 &= -\omega_0 x_2 + b_1 \varphi(x_1 + c^* x_3) \\ \dot{x}_2 &= \omega_0 x_1 + b_2 \varphi(x_1 + c^* x_3) \\ \dot{x}_3 &= Ax_3 + b\varphi(x_1 + c^* x_3). \end{aligned} \tag{2}$$

Here A is a constant $(n-2) \times (n-2)$-matrix, all eigenvalues of which have negative real parts, b and c are $(n-2)$-dimensional vectors, b_1 and b_2 are certain numbers.

Combined application of the harmonic linearization method, the classical method of small parameter, and numerical methods permit us to compute periodic oscillations of certain multistage procedure, where at the first step the harmonic linearization method is applied.

In the basic, noncritical, case we suppose that the relation $\varphi(\sigma) = \varepsilon\psi(\sigma)$, where ε is a small parameter, is satisfied. In the sequel, without loss of generality, we can assume that for A there exists the number $\alpha > 0$ such that

$$x_3^*(A + A^*)x_3 \le -2\alpha |x_3|^2, \ \forall x_3 \in R^{n-2}.$$

We introduce the function

$$K(a) = \int_0^{2\pi/\omega_0} \psi(\cos(\omega_0 t)a)\cos(\omega_0 t)dt.$$

Theorem 1. *If the conditions*

$$K(a) = 0, \quad b_1 \frac{dK(a)}{da} < 0,$$

are satisfied, then for sufficiently small $\varepsilon > 0$ system (1) has T-periodic solution such that

$$\begin{aligned} r^*x(t) &= a\cos(\omega_0 t) + O(\varepsilon), \\ T &= \tfrac{2\pi}{\omega_0} + O(\varepsilon). \end{aligned}$$

This periodic solution is stable in the sense that there exists its certain ε-neighbourhood such that all solutions with the initial data from this ε-neighbourhood remain in it in increasing time t.

The described in Theorem 1 "standard", basic, method of harmonic linearization turns out too rough to locate periodic oscillations in nonlinear systems, satisfying the generalized Routh-Hurwitz conditions. The relevance of such problem is stimulated by Aizerman's conjecture [14, 16].

The extension of Theorem 1 in the spirit of classical research of critical cases in the theory of motion stability makes it possible to obtain effective estimates for periodic oscillations in systems, satisfying the generalized Routh-Hurwitz conditions.

Consider a class of the functions $\varphi(\sigma)$ of the form

$$\begin{aligned} \varphi(\sigma) &= \mu\sigma, \quad \forall \sigma \in (-\varepsilon, \varepsilon), \\ \varphi(\sigma) &= M\varepsilon^3, \quad \forall \sigma > \varepsilon, \\ \varphi(\sigma) &= -M\varepsilon^3, \quad \forall \sigma < -\varepsilon, \end{aligned} \tag{3}$$

where μ and M are certain positive numbers, ε is a small positive parameter.

Similar classes of functions were considered before in studying Aizerman's conjecture [14, 16].

Theorem 2. *If the inequalities $b_1 < 0$, $\mu b_2(c^*b + b_1) + b_1\omega_0 > 0$, are satisfied, then system (2) with nonlinearity (3) has T-periodic solution with the initial data*

$$\begin{aligned} x_1(0) &= O(\varepsilon^2), \quad x_3(0) = O(\varepsilon^2), \\ x_2(0) &= -\sqrt{\frac{\mu(\mu b_2(c^*b + b_1) + b_1\omega_0)}{3\omega_0 M(-b_1)}} + O(\varepsilon). \end{aligned}$$

and

$$T = \frac{2\pi}{\omega_0} + O(\varepsilon).$$

The described in Theorems 1 and 2 periodic solution can be considered as certain "support" (basic) periodic oscillations and system (1) with the considered above nonlinearities as "generating" start system in the algorithms of seeking the periodic solutions of another system, namely

$$\frac{dx}{dt} = P_0 x + q f(r^* x), \tag{4}$$

In this case, we can organize a finite sequence of the functions $\varphi_j(\sigma) \quad j = 1, \ldots, m$, such that the graphs of each pair φ_j and φ_{j+1} are close to each other and $\varphi_1(\sigma) = \varphi(\sigma)$, $\varphi_m(\sigma) = f(\sigma)$. Then, for the system

$$\frac{dx}{dt} = Px + q\varphi_j(r^* x) \tag{5}$$

with $\varphi_1(\sigma) = \varphi(\sigma)$ and small ε we take the periodic solution $g_1(t)$, described in either Theorem 1 or Theorem 2. Two cases occur: either all points of this periodic solution are situated in a domain of attraction of the stable periodic solution $g_2(t)$ of system (5) with $j = 2$, or in passing from system (5) with $j = 1$ to system (5) with $j = 2$ we have a bifurcation of stability loss and a vanishing of periodic solution.

In the first case, we can numerically find $g_2(t)$ when the trajectory of system (5) with $j = 2$ begins at the initial point $x(0) = g_1(0)$.

Starting from the point $g_1(0)$, after transient process the computational procedure outputs into the periodic solution $g_2(t)$ and calculates it. For this purpose the interval $[0, \tau]$, on which the computation occurs, must be sufficiently large.

After computation of $g_2(t)$ it is possible to go to the following system (5) with $j = 3$ and to organize a similar procedure for computation of the periodic solution $g_3(t)$ when a trajectory, which in increasing t tends to the periodic trajectory $g_3(t)$, starts from the initial point $x(0) = g_2(\tau)$.

Proceeding then this procedure for sequential computing $g_j(t)$ and making use of trajectories of system (5) with the initial data $x(0) = g_{j-1}(\tau)$, we arrive by numerical computation of periodic solution of system (4) or observe, at a certain step, a bifurcation of stability loss and a vanishing of periodic solution.

We give two examples of applying this procedure.

Example 1. Consider system (2) with the function $\varphi(\sigma) = \varepsilon\psi(\sigma)$, $\psi(\sigma) = k_1\sigma + k_3\sigma^3$. Then we have

$$K(a) = (k_1 a + \frac{3}{4}k_3 a^3)\frac{\pi}{\omega_0}.$$

It follows that a can be determined from the equation $K(a) = 0$ in such way

$$a = a_1 = \sqrt{-\frac{4k_1}{3k_3}},$$

and a stability condition takes the form $-b_1 k_1 < 0$.

Let be $k_1 = -3, k_3 = 4$, $\omega_0 = 1, b_1 = -1, b_2 = 1, A = -1, c = 1, b = 1$. Then $a_1 = 1$.

Using the classical harmonic linearization method [6], we obtain that for any $\varepsilon > 0$, system (2) has a periodic solution and in this case $\sigma(t) = r^* x(t) \approx \cos t$.

By Theorem 1 for small $\varepsilon > 0$, system (2) has a periodic solution of the form

$$\begin{aligned} x_1(t) &= \cos t + O(\varepsilon) \\ x_2(t) &= \sin t + O(\varepsilon) \\ x_3(t) &= O(\varepsilon). \end{aligned}$$

Further, using the above computational procedure, we obtain a periodic solution of system (2) for $\varphi_j(\sigma) = \varepsilon_j\varphi(\sigma)$, $\varepsilon_1 = 0,1, \varepsilon_2 = 0,3, \varepsilon_3 = 0,6, \varepsilon_4 = 0,7, \varepsilon_5 = 0,9, \varepsilon_6 = 1$. In Figure 3 it is shown a projection of periodic solution thus computed on the plane $\{x_1, x_2\}$ when $\varepsilon = 1$. For these periodic solution the graph $\sigma(t) = x_1(t) + x_3(t)$ is represented. In this case for $\varepsilon = 1$, the output $\sigma(t)$ is substantially not harmonic and the filter hypothesis is untrue. Therefore, here it is impossible, in principle, to justify a standard harmonic linearization method, founding on the filter hypothesis.

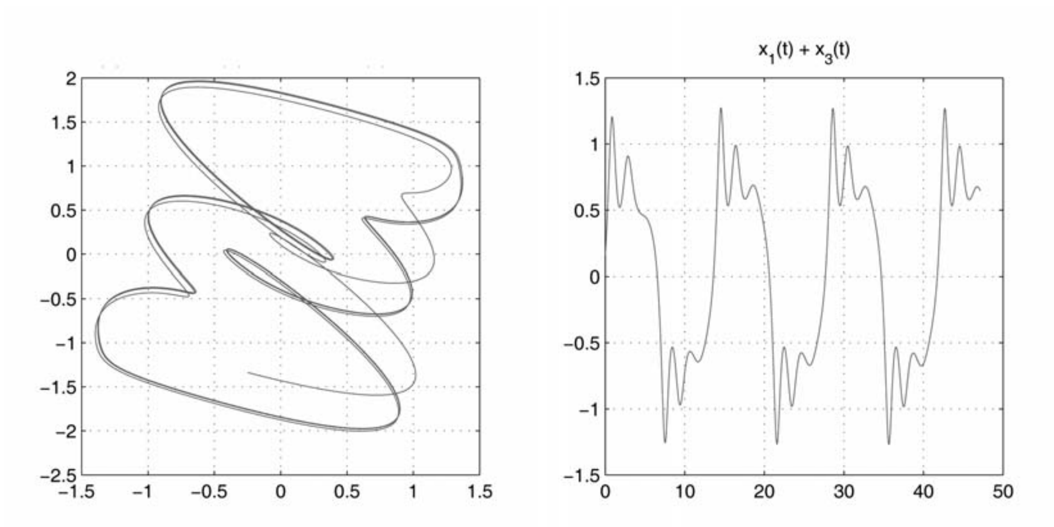

Figure 3: $\varepsilon = 1$.

Example 2.

Consider system (2) with the function

$$W(p) = \frac{p-1}{p^2+1} + \frac{1}{p+1}.$$

Here $r^*q = -1$, $\omega_0 = 1$, $b_1 = -1$, $b_2 = -1$, $A = -1$, $c = 1$, $b = -1$.

If $\varphi_j(\sigma) = k\sigma$ then system (2) is stable for all $k \in (0, +\infty)$.

Let

$$\varphi_j(\sigma) = \mu\sigma, \quad \forall\sigma \in (-\varepsilon_j, \varepsilon_j),$$
$$\varphi_j(\sigma) = M\varepsilon_j^3, \quad \forall\sigma > \varepsilon_j,$$
$$\varphi_j(\sigma) = -M\varepsilon_j^3, \quad \forall\sigma < -\varepsilon_j.$$

Here $\mu = 2, M = 1$ and ε_j – positive parameters.

By theorem 2 initial conditions for periodic solution are the following

$$x_1(0) = O(\varepsilon), x_3(0) = O(\varepsilon), x_2(0) = -\sqrt{\frac{2}{3}} + O(\varepsilon).$$

Further, using the above computational procedure, we obtain a periodic solution of system (2) for $\varepsilon_1 = 0, 1, ..., \varepsilon_7 = 0, 7$. In Figure 4 it is shown a projection of periodic solution thus computed on the plane $\{x_1, x_2\}$ when $\varepsilon = 0.7$.

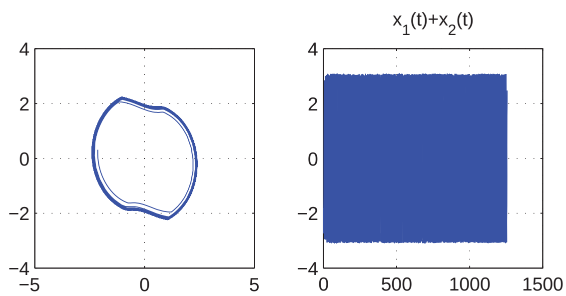

Figure 4: $\varepsilon = 0.7$.

For $\varepsilon = 0.8$ the periodic solution is destroyed.

4 The circle criterion of nonexistence of periodic oscillations with a given period for discrete systems

Consider a discrete system

$$x(t+1) = Px(t) + q\varphi(t, r^* x(t)), \qquad (6)$$
$$x \in \mathbb{R}^n, \qquad t \in \mathbb{Z}.$$

Here q and r are constant n-dimensional vectors, P is a constant real $n\times n$-matrix, $\varphi(t,\sigma)$ is a scalar function, satisfying on a certain set $\Omega \subset \mathbb{R}^1$ the following condition: for any $t \in \mathbb{Z}$ and $\sigma \in \Omega \setminus \{0\}$ the relation

$$k_1\sigma^2 < \varphi(t,\sigma)\sigma < k_2\sigma^2. \qquad (7)$$

is valid. If $0 \in \Omega$, then we require additionally $\varphi(\cdot,0) = 0$.

Introduce *the transfer function* $W : \mathbb{C} \to \mathbb{C}$ of system (6):

$$W(p) = r^*(P - pI)^{-1}q, \qquad p \in \mathbb{C}. \qquad (8)$$

Theorem (the circle criterion). *Suppose, for a certain natural N the following inequalities*

$$Re\left[\left(1+k_1W\left(e^{\frac{2\pi j}{N}i}\right)\right)^*\left(1+k_2W\left(e^{\frac{2\pi j}{N}i}\right)\right)\right] \geq 0, \qquad (9)$$
$$\forall j = 0,\ldots,N-1$$

are satisfied.

Then there does not exist N-periodic sequence of the vectors $x(t)$, which satisfies system (6) with the condition on nonlinearity (7) and the inclusion

$$c^*x(t) \in \Omega, \qquad \forall t \in \mathbb{Z}. \qquad (10)$$

The proof of this theorem is similar to the proof of Theorem 13.1 from [13] and is based on the properties of discrete Fourier transformation.

We give two applications of the circle criterion.

Example 3. Logistic map

A discrete logistic equation is as follows

$$x(t+1) = \mu x(t)(1 - x(t)), \qquad (11)$$
$$\mu \in \mathbb{R}_+, \qquad x(0) \in (0;1).$$

For all $t \geq 2$ we have the obvious restriction

$$x(t) \in \left[\frac{\mu^2(4-\mu)}{16};\frac{\mu}{4}\right] \qquad (12)$$

From (11) we obtain $x(t+1) = \mu x(t) - \mu x(t)^2$. Make the change of variables: $\sigma(t) = x(t) - a$, where $a = 1 - \frac{1}{\mu}$.

Now the system takes the form

$$\sigma(t+1) = \mu\sigma(t) - \left[\mu(\sigma(t)+a)^2 - (\mu-1)a\right] \qquad (13)$$

This system has the form (6), where $P = \mu$, $q = -1$,
$r = 1, \varphi(t, \sigma) = \varphi(\sigma) = \left[\mu(\sigma+a)^2 - (\mu-1)a\right]$.

A transfer function of such system is as follows

$$W(p) = \frac{1}{p-\mu},$$

and restriction (12) becomes

$$\sigma(t) \in \left[\frac{\mu^2(4-\mu)}{16} - a;\ \frac{\mu}{4} - a\right] = [\sigma_{min};\ \sigma_{max}]. \qquad (14)$$

Note that $\varphi(0) = 0$, and the nonlinearity φ on the set $\Omega = [\sigma_{min}; \sigma_{max}]$ lies inside the sector bounded by two straight lines: $\xi = k_1\sigma$ and $\xi = k_2\sigma$, where

$$k_1 = \frac{\varphi(\sigma_{min})}{\sigma_{min}}, \qquad k_2 = \frac{\varphi(\sigma_{max})}{\sigma_{max}}.$$

For $\mu < 3$, all trajectories of system (13) tend to a unique equilibrium. Now we consider $\mu \in (3,\ 4)$.

Consider a criterion for $N = 3$.

For $j = 0$, condition (9) is the inequality

$$\left(1 - \frac{k_1}{\mu - 1}\right)\left(1 - \frac{k_2}{\mu - 1}\right) \geq 0,$$

which is satisfied for $\mu \in (2;\ 4)$.

For $j = 1, 2$ inequalities (9) have the form

$$1 + (k_1 + k_2)\left(-\frac{\mu + \frac{1}{2}}{\mu^2 + \mu + 1}\right) +$$

$$+k_1k_2\left(\frac{1}{\mu^2 + \mu + 1}\right) \geq 0,$$

what is equivalent to the relation

$$\mu^2 + \mu + 1 - (k_1 + k_2)\left(\mu + \frac{1}{2}\right) \geq 0$$

This inequality is satisfied for $\mu < 3.6395$. Thus, we obtain that $\mu < 3.6395$ the shifted logistic map, and therefore, original logistic map cannot have a cycle of period 3. This estimation is close to parameter μ which correspond to existence [13] of solutions with period 3: $\mu = 1 + \sqrt{8} = 3,8284$.

Example 4. Henon map

Consider a *Henon map* with one real parameter

$$\begin{cases} x(t+1) &= 1 + y(t) - ax(t)^2 \\ y(t+1) &= 0.3x(t) \end{cases} \tag{15}$$

From (15) we obtain

$$x(t+1) = 1 + 0.3x(t-1) - ax(t).$$

Having performed the changes $u(t) = x(t) - \Delta$ and letting $v(t+1) = u(t)$, we obtain the new two-dimensional map

$$\begin{cases} u(t+1) &= 0.3v(t) + 1 - 0.7\Delta - a(u(t) + \Delta)^2 \\ v(t+1) &= u(t) \end{cases}$$

The above system of "offset" Henon map is a system of the form (6) with

$$P = \begin{pmatrix} 0 & 0.3 \\ 1 & 0 \end{pmatrix},\ q = \begin{pmatrix} -1 \\ 0 \end{pmatrix},\ r = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$\varphi(t, \sigma) = \varphi(\sigma) = a(\sigma + \Delta)^2 + 0.7\Delta - 1,$$
$$\sigma = c^*(u, v)^* = u.$$

Transfer function of above system is

$$W(p) = \frac{p}{p^2 - 0.3}.$$

Let, as in the previous example, $N = 3$. Substituting the values for circle criterion, we obtain

$$p_0 = e^{\frac{2\pi j}{3}i} = 1 \implies W(p_0) = \frac{10}{7};$$

$$p_{1,2} = e^{\frac{2\pi j}{3}i} \implies W(p_{1,2}) = \frac{-35 \mp 65\sqrt{3}i}{139}$$

Find Δ from the conditions $\varphi(\Delta) = 0$ and $\Delta > 0$. Simple calculation gives

$$\Delta = \frac{\sqrt{49+400a}-0.7}{20a}.$$

Numerical modeling shows that for $t \to +\infty$ the number σ lies in the interval

$$\sigma \in (-1.1a-0.4,\ 0.1a+0.52) = (\sigma_{min},\ \sigma_{max})$$

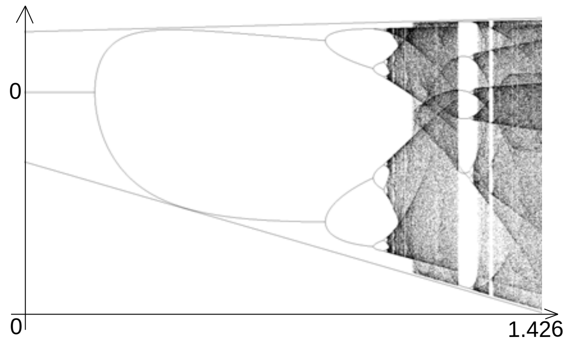

Figure 5: Bifurcations diagram of offset Henon map and the considered restrictions of attractor.

The nonlinearity now can be collected in the sector:

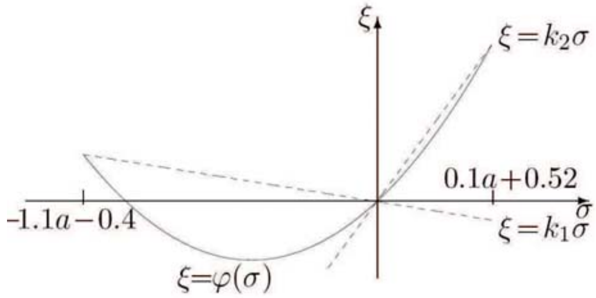

Figure 6: Nonlinearity of offset Henon map.

Here

$$k_1 = \frac{\varphi(\sigma_{min})}{\sigma_{min}} = -1.1a^2 - 0.4a + \sqrt{0.49+4a} - 0.7,$$

$$k_2 = \frac{\varphi(\sigma_{max})}{\sigma_{max}} = 0.1a^2 + 0.52a + \sqrt{0.49+4a} - 0.7.$$

For these k_1 and k_2, find a for which the circle criterion is valid.

1) $j = 0$

$$W(p_0) = \frac{10}{7}.$$

The inequality from the circle criterion becomes

$$Re[(1+\frac{10}{7}k_1)(1+\frac{10}{7}k_2)] \geq 0,$$

what is equivalent to

$$(11a^2+4a-\sqrt{49+400a})\cdot$$
$$\cdot(a^2+5.2a+\sqrt{49+400a}) \leq 0$$

The elementary analysis shows that this inequality is satisfied for $a < 1.3$.

2) $j = 1,2$

$$W(p_{1,2}) = \frac{-35 \mp 65\sqrt{3}i}{139}.$$

Then (9) takes the form of the following inequality

$$1 - \frac{35}{139}(k_1 + k_2) + \frac{100}{139}k_1 k_2 \geq 0,$$

which is satisfied for $a < 1.129$.

Thus, for $a < 1.129$ the shifted Henon map, and therefore, the original Henon map cannot have a cycle of period 3.

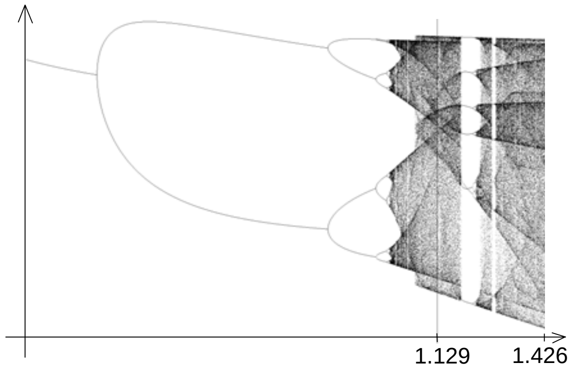

Figure 7: The obtained estimate on the bifurcational diagram of Henon map.

5 Feigenbaum effect for nonunimodal map

Discrete Phase-Locked Loops with sinusoidal characteristic of phase discriminator are described in details in [17].

In case of initial frequency of master and local generators coincidence equation is given by

$$\sigma(x+1) = \sigma(x) - r\sin(\sigma(x)) \tag{16}$$

where r is a positive number.

In the system (16) there is transition to chaos via the sequence of period doubling bifurcations.

Equation (16) is not unimodal map [3, 9, 23], so we can not apply the usual Renorm-Group method for its analytical investigation. Some first bifurcation parameters can be calculated analytically [17], the others can be found only by means of numerical calculations.

The first 13 calculated bifurcation parameters of period doubling bifurcation of (16)

$r_1 = 2$	$r_2 = 3.141592653589790$
$r_3 = 3.445229223301312$	$r_4 = 3.512892457411257$
$r_5 = 3.527525366711579$	$r_6 = 3.530665376391086$
$r_7 = 3.531338162105000$	$r_8 = 3.531482265584890$
$r_9 = 3.531513128976555$	$r_{10} = 3.531519739097210$
$r_{11} = 3.531521154835959$	$r_{12} = 3.531521458080261$
$r_{13} = 3.531521523045159$	

Here r_2 is bifurcation of splitting global stable cycle of period 2 into two local stable cycles of period 2. The other r_j correspond to period doubling bifurcations.

We have here following Feigenbaum's effects:

$$\delta_2 = (r_2 - r_1)/(r_3 - r_2) = 3.759733732581654$$

$$\delta_3 = \frac{r_3 - r_2}{r_4 - r_3} = 4.487467584214882$$

$$\delta_4 = \frac{r_4 - r_3}{r_5 - r_4} = 4.624045206680584$$

$$\delta_5 = \frac{r_5 - r_4}{r_6 - r_5} = 4.660147831971297$$

$$\delta_6 = \frac{r_6 - r_5}{r_7 - r_6} = 4.667176508904449$$

$$\delta_7 = \frac{r_7 - r_6}{r_8 - r_7} = 4.668767988303247$$

$$\delta_8 = \frac{r_8 - r_7}{r_9 - r_8} = 4.669074658227896$$

$$\delta_9 = \frac{r_9 - r_8}{r_{10} - r_9} = 4.669111696537520$$

$$\delta_{10} = \frac{r_{10} - r_9}{r_{11} - r_{10}} = 4.669025736544542$$

$$\delta_{11} = \frac{r_{11} - r_{10}}{r_{12} - r_{11}} = 4.668640891299296$$

$$\delta_{12} = \frac{r_{12} - r_{11}}{r_{13} - r_{12}} = 4.667817727564633$$

So here we consider approximation of Feigenbaum's constant for considered nonunimodal map.

6 Conclusion

Difficult problems on a search of periodic solutions stimulated a synthesis development of analytical-numerical and numerical methods. Some of these directions are considered in the present paper.

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