Free Analytical Vibrations of Smart Beams using Distributed Transfer Function Analysis

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Introduction. The modal Effective Electromechanical Coupling Coefficient (EMCC) is known as a critical parameter in justifying the performance of piezoelectric materials since it describes the efficiency of converting mechanical strain to electric charges and vice versa. For a vibrating structure, the EMCC is defined by [1]

$$k_r^2 = \frac{(\omega_r^{oc})^2 - (\omega_r^{sc})^2}{(\omega_r^{sc})^2} \tag{1}$$

Where ω^{oc} and ω^{sc} are the natural frequencies of the structure under the open-circuit (OC) and short-circuit (SC) boundary conditions for the rth mode. The cantilever beam under consideration, Figure 1, assumes no losses and perfect bonding between the piezoelectric layer and the host elastic beam.

Equations of Motion. Consider a thin elastic beam divided into three regions as shown in Figure 2. The first and last regions are made only of an elastic material while the second one additionally contains a piezoelectric layer completely covering the elastic one. The Euler-Bernoulli in-plane strain is given by

$$\varepsilon_1 = u_x - z w_{xx} \tag{2}$$

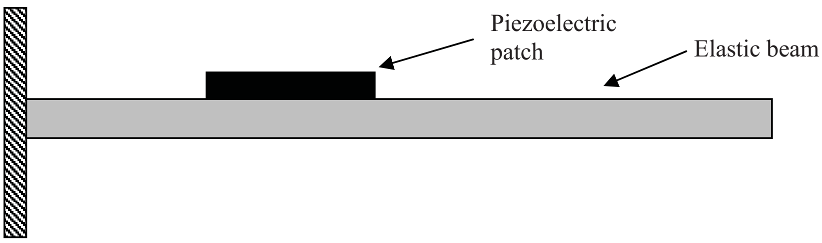

Figure 1. Smart cantilever beam

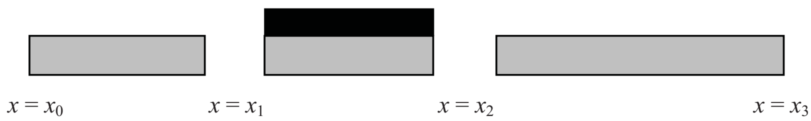

Figure 2. Beam segmentation

Where u and w are the in-plane and transverse deformation, respectively. The constitutive equations for the piezoelectric beam are

$$\begin{aligned}\sigma_1 &= C_{11}^E \varepsilon_1 - e_{31} E_3 \\ D_3 &= e_{31}\varepsilon_1 + \in_{33}^{\varepsilon} E_3\end{aligned} \tag{3}$$

Where σ, E and D denote, respectively, the stress, electric field and electric displacement while, e, C^E and $\in^{\varepsilon}$ stand for the piezoelectric, elastic, and dielectric constants. Applying Hamilton's principle on the second region results in the following set of mechanical equations

$$\begin{aligned}m\ddot{u} - k u_{xx} &= 0 \\ m\ddot{w} + d w_{xxxx} &= 0\end{aligned} \tag{4}$$

$$m = \rho_b A_b + \rho_p A_p, \quad k = E_b A_b + E_p A_p, \quad d = E_b \bar{I}_b + E_p \bar{I}_p$$

With ρ, A and $\bar{I}$ denoting the density, cross section area and moment of inertia about the neutral axis, respectively. Also, subscripts b and p represent quantities for the beam layer and the piezoelectric layer, respectively. The third resulting electromechanical equation states that

$$\int_{x_1}^{x_2} (-e_{31} t_p u_x + e_{31}\alpha w_{xx} + \in_{33}^{\varepsilon} V/t_p)\, dx = -\int_{\Omega} q\, d\Omega \tag{5}$$

Where V is the potential difference of the piezoelectric electrodes, q is a point charge on the piezoelectric surface Ω, and α is the distance from the neutral axis of the two layers to the mid-plane of the piezoelectric layer. The resulting boundary conditions of the piezoelectric layer at x_1 or x_2 are

$$\begin{aligned}&k u_x + e_{31} A_p V/t_p = 0 \quad \text{or} \quad u = 0 \\ &d w_{xx} - \alpha e_{31} A_p V/t_p = 0 \quad \text{or} \quad w_x = 0 \\ &d w_{xxx} = 0 \quad \text{or} \quad w = 0\end{aligned} \tag{6}$$

Which represent the essential and natural boundary conditions required for applying the displacement and force continuity between the three regions.

Distributed Transfer Function. The equations of motion are first transformed to the Laplace domain. The transformed equations are cast into a state space form where the state vector is made of the deformation vector $D(x,s)$, and the strain vector $P(x,s)$ as follows [2,3]

$$Y_i(x,s) = \begin{Bmatrix} D(x,s) \\ P_i(x,s) \end{Bmatrix}, \tag{7}$$

$$D(x,s) = \left\{ u_i, \quad w_i, \quad \frac{\partial w_i}{\partial x} \right\}^T, \qquad P_i(x,s) = \left\{ \frac{\partial u_i}{\partial x}, \quad \frac{\partial^2 w_i}{\partial x^2}, \quad \frac{\partial^3 w_i}{\partial x^3} \right\}^T$$

The state space form is then given by

$$\frac{\partial}{\partial x} Y_i(x,s) = F_i(s) Y_i(x,s), \tag{8}$$

$$F_i(s)=\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ m_i s^2/k_i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -m_i s^2/d_i & 0 & 0 & 0 & 0 \end{bmatrix} \tag{9}$$

with $m_1 = m_3 = m$, $m_2 = m_b$, $k_1 = k_3 = k$, $k_2 = k_b$, $d_1 = d_3 = d$, $d_2 = d_b$

The essential displacement continuity conditions are cast in the following matrix equations

$$M_i Y_i(x_{i-1},s) + N_i Y_i(x_i,s) = \gamma_i(s) \qquad i = 1,2,3$$

$$M_i=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\quad N_1 = N_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\quad N_3=\begin{bmatrix} k & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & d \end{bmatrix} \tag{10}$$

$$\gamma_1(s)=\begin{Bmatrix} 0_{3\times1} \\ D(x_1,s) \end{Bmatrix},\qquad \gamma_2(s)=\begin{Bmatrix} D(x_1,s) \\ D(x_2,s) \end{Bmatrix},\qquad \gamma_3(s)=\begin{Bmatrix} D(x_2,s) \\ 0_{3\times1} \end{Bmatrix}$$

Similarly, the force continuity conditions are shown to be

$$\begin{aligned} B_1 Q_1(x_1,s) &= B_2 Q_2(x_1,s) + \bar{V} \\ B_2 Q_2(x_2,s) + \bar{V} &= B_3 Q_3(x_2,s) \end{aligned} \tag{11}$$

$$B_1 = B_3 = \begin{bmatrix} k_b & 0 & 0 \\ 0 & d_b & 0 \\ 0 & 0 & d_b \end{bmatrix},\qquad B_2 = N_3,\qquad \bar{V} = \frac{e_{31} A_p V}{t_p}\begin{Bmatrix} 1 \\ -\alpha \\ 0 \end{Bmatrix}$$

The solution of the state space equation for each region is given by [3]

$$Y_i(x,s) = \Psi_i(x,s)\gamma_i(s) = \begin{bmatrix} \Psi_i^{11} & \Psi_i^{12} \\ \Psi_i^{21} & \Psi_i^{22} \end{bmatrix}\gamma_i(s) \tag{12}$$

with $\Psi_i(x,s) = e^{F_i x}\left[M_i e^{F_i x_{i-1}} + N_i e^{F_i x_i}\right]^{-1}$.

At the common interfaces between the three regions, the following equations hold true

$$Y_1(x_1,s) = Y_2(x_1,s)\,,\qquad Y_2(x_2,s) = Y_2(x_2,s) \tag{13}$$

substituting equation (11) in the last one eliminates Q_i ($i = 1,2,3$) and the result is the dynamic matrix equation of motion

$$K(s)\gamma_2(s) = \bar{F} \tag{14}$$

with

$$K(s)=\begin{bmatrix} B_1\Psi_1^{22}(x_1) - B_1\Psi_2^{21}(x_1) & -B_2\Psi_2^{22}(x_1) \\ -B_2\Psi_2^{21}(x_2) & B_3\Psi_3^{21}(x_2) - B_2\Psi_2^{22}(x_2) \end{bmatrix},\qquad \bar{F}=\begin{Bmatrix} \bar{V} \\ \bar{V} \end{Bmatrix} \tag{15}$$

Electrical Boundary Conditions. The electrical boundary condition for the SC condition is $V = 0$ and this reduces equation (13) to

$$K(s)\gamma_2(s) = 0 \tag{16}$$

In OC condition, the total surface charge $\int_\Omega q d\Omega$ vanishes which reduces equation (5) to

$$V = \frac{e_{31} t_p}{\in_{33}^{\varepsilon} (x_2 - x_1)} \left[u_x(x_2) - u_x(x_1) - \alpha\left(w_{xx}(x_2) - w_{xx}(x_1) \right) \right] \tag{17}$$

As a result, the last equation is substituted back in (13) resulting in

$$\left[K(s) + \overline{B} \right] \gamma_2(s) = 0 \tag{18}$$

$$\overline{B} = \begin{bmatrix} B_{oc} & -B_{oc} \\ B_{oc} & -B_{oc} \end{bmatrix}, \qquad B_{oc} = \begin{bmatrix} a & 0 & -a\alpha \\ -a\alpha & d0 & a\alpha^2 \\ 0 & 0 & 0 \end{bmatrix}, \qquad a = \frac{e_{31}^2 t_p}{\in_{33}^{\varepsilon} (x_2 - x_1)}$$

The SC and OC frequencies are the values that force the determinant of the dynamic transfer matrix to vanish in both cases. Noting that $s = j\omega$ ($j = \sqrt{-1}$), the characteristics equations for the SC and OC conditions can be written, respectively, as

$$\det\left[K(j\omega) \right] = 0 \ , \quad \det\left[K(j\omega) + \overline{B} \right] = 0 \tag{19}$$

Results. Preliminary results have shown that the natural frequencies obtained by the current DTF method exactly match the ones obtained by the classical differential equation solution.

Conclusion. Unlike the classical boundary value problem approaches which get complicated for complex systems, the current modeling approach is systematically simple and requires no previous assumption of the solution type. Noting that the definition of EMCC is very sensitive to frequency predictions, the DTF method under consideration is of special interest in the design of smart structures since the exact frequencies can be easily calculated, leading to exact determination of the modal EMCC.

References

[1] S. Basak, A. Raman and S. Garimella, 2005, " Dynamic response optimization of piezoelectrically excited thin resonant beams", *Journal of Vibration and Acoustics*, **127**, pp.18-27

[2] B. Yang, 1994, “Distributed transfer function analysis of complex distributed parameter systems” *Journal of Applied Mechanics,* **61**, pp.84-92.

[3] Y. Liu and K. Wang, 2000, “Distribution of active and passive constraining sections for hybrid constrained layer damping treatments”, *Journal of Intelligent Material Systems and Structures*, **13**, pp. 23-34.