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MODELING AND VARIATIONAL ANALYSIS OF CONTROL PROBLEMS FOR ELASTIC BODY MOTIONS

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Abstract. Modeling problems for controlled motions of an elastic body is considered. A variational principle, in which displacement and stress fields are varied, is proposed based on the method of integrodifferential relations and the linear theory of elasticity. A regular numerical algorithm of constrained minimization for the initial-boundary value problems is worked out. The algorithm allows us to estimate explicitly the local and integral quality of numerical solutions obtained. As an example, the problem of lateral controlled motions for a 3D rectilinear elastic prism with a rectangular cross section is investigated.

1 Introduction

Variational principles in mechanics involving the theory of elasticity have been very thoroughly developed and intensively studied by scientists. Among these formulations the minimum principles for potential and complementary energy, the Hamilton principle can be mentioned (see [22], for example). Many different approaches to deduce variational principles for mechanical problems are presented in the literature. It can be noted the recent publication concerning the semi-inverse method suggested by He [7]. The variational formulation of the finite element method is broadly used in scientific and engineering applications. The mathematical origin of the method can be traced to a paper by Courant [5]. Other numerical approaches, e.g., Petrov-Galerkin method [3], [4] or the least squares method [21], are being actively developed for solid mechanics. In all these methods it is supposed that some of the elasticity relations (equilibrium equations, boundary conditions in terms of stresses and etc.) are generalized and the exact solution is approximated by a finite set of trial functions.

Elastic properties of structure elements can essentially affect the dynamical behavior of the whole system. Some parts of mechanical structures with distributed parameters are modeled as elastic bodies with given stiffness and inertia characteristics. A significant number of numerical methods has been developed for modeling the behavior of dynamical systems described by initial-boundary value problems. One of the most widespread approaches to solving problems of such kind is the method of separation of variables [6]. In [1] a regular perturbation method (a small parameter method) for investigating the dynamics of weakly non-uniform thin rods with arbitrary distributed loads and different boundary conditions was proposed. Based on the classical Rayleigh-Ritz approach, a numerical analytical method of fast convergence was developed in [2] to find precise values of unknown functions for arbitrary distributed stiffness and inertia characteristics of elastic systems. In modeling the elastic systems the methods of finite-dimensional approximations, for example the decomposition method and the regularization method, were developed to reduce an initial-boundary value problem for partial differential equations to a system of ordinary differential equations [8], [9]. The direct discretization methods in optimal control problems are also well known (see, e.g. [10]).

The aim of this contribution is to develop the method of integrodifferential relations (MIDR) for dynamical linear elasticity problems based on the integral formulation of the constitutive equations and to apply this approach to analysis and optimization of 3D elastic body behavior. The basic ideas of MIDR were proposed and discussed by Kostin and Saurin [11]–[19].

In the next section, the statement of a dynamic linear elasticity problem is discussed. In the third section the method of integrodifferential relations and a new variational formulation of the initial boundary problem in displacements and stresses are considered. The stationary conditions equivalent to the constitutive relations are obtained. In Section 4 a numerical algorithm used the finite approximations of unknown functions (displacements and stresses) is developed [12] and the effective integral and local bilateral estimates of solution quality are obtained relying on the extremal properties of the finite dimensional variational problem. Sections 5 and 6 are devoted to numerical modeling, optimization, and analysis of controlled motions of a 3D elastic beam. Concluding remarks are given in Section 7.

2 Statement of the problem

Consider an elastic body occupying a bounded domain Ω with an external piecewise smooth boundary γ. Taking into account the assumption of the linear theory of elasticity about smallness of elastic deformations and relative velocities the motion of the body can be described by the following system of partial differential equations [16]:

$$\boldsymbol{\sigma} = \mathbf{C} : \boldsymbol{\varepsilon}, \quad \mathbf{p} = \rho \mathbf{u}^{\bullet}, \tag{1}$$

$$\nabla \cdot \boldsymbol{\sigma} - \mathbf{p}^{\bullet} + \mathbf{f} = 0, \quad \boldsymbol{\varepsilon} = \frac{1}{2}\left(\nabla \mathbf{u} + \nabla \mathbf{u}^{\mathrm{T}}\right). \tag{2}$$

Here $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ are the stress and strain tensors, **p** and **u** are the momentum density and displacement vectors, **f** is the vector of volume forces, **C** is the elastic modulus tensor, and ρ is the volume density of the body. The dots above the symbols denote partial time derivatives, and $\nabla = (\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3)^{\mathrm{T}}$ is the gradient operator in the Cratesian coordinate space $x = \{x_1, x_2, x_3\}$. The dots and colons between vectors and tensors point out to their scalar and double scalar products, respectively. The components C_{ijkl} of the tensor **C** are characterized by the following symmetry property: $C_{ijkl} = C_{ijlk} = C_{klij}$. The superscript T denotes the transposition operator.

Let us constrain ourselves to the case of the linear boundary conditions expressed componentwise in the form:

$$\alpha_k(x)u_k + \beta_k(x)q_k = v_k, \quad x \in \gamma; \quad \mathbf{q} = \boldsymbol{\sigma} \cdot \mathbf{n}, \quad k = 1, 2, 3, \tag{3}$$

where $\mathbf{q}$ is the loading vector, $\mathbf{n}$ is the unit vector pointing in the direction of the outward normal to the boundary γ, u_k and q_k are the components of the vector functions **u** and **q**, α_k and β_k are given coordinate functions defining the type of boundary conditions. In particular, if $\alpha_k = 1$ and $\beta_k = 0$ on some part of the boundary γ then, according to Eq. (3), the displacement u_k is set by the boundary vector function $\mathbf{v}$ via its component v_k. To the contrary, if $\alpha_k = 0$ and $\beta_k = 1$ then the external load component q_k are defined through v_k on γ. Conditions (3) include also various combinations of linear elastic supports if the pair α_k, β_k is simultaneously nonzero on a certain part of the boundary. The components of boundary vector $\mathbf{v}$ are given functions of the time t and coordinates x.

It is supposed that at the initial instant $t = 0$ the distribution of displacements **u** and momentum density **p** are given as sufficiently smooth functions of the coordinates x

$$\mathbf{u}(0, x) = \mathbf{u}^0(x), \quad \mathbf{p}(0, x) = \mathbf{p}^0(x), \quad x \in \Omega. \tag{4}$$

Note that initial conditions (4) and boundary conditions (3) should be consistent [14].

3 The method of integrodifferential relations

Relations (1)–(4) describe the deformed state of an elastic body at any internal point x and any instant t. In addition, it is assumed that the stresses and displacements at the internal points of the body should tend to boundary stresses and displacements, i.e. conditions (3) are satisfied. It is implied that the components of the elastic modulus tensor $\mathbf{C}$ and mass density ρ, defined in the interior of the domain Ω, continuously pass through the boundary γ. Analogously, the displacement and momentum density vectors tend continuously to their initial values given by relations (4). On the other hand, it is necessary to take into account that boundary and initial conditions (3), (4) are generated by specific physical and geometrical factors. For example, some part of the boundary could be an interface between two or more media (elastic or inelastic). In this case, any boundary point belongs simultaneously to the body under consideration and the bodies which generate these boundary conditions. So such points on the boundary belong simultaneously to material parts with different mechanical properties. In other words, the elastic modulus tensor $\mathbf{C}$ and mass density function ρ on such surfaces, strictly speaking, are not defined.

To introduce these uncertainties into the dynamical linear elasticity problem a combined integral relation instead of stress-strain and velocity-momentum relations (1) was proposed in [12], and the following integrodifferential formulation of initial-boundary value problem (1)–(4) was given: to find such functions $\mathbf{u}^*$, $\boldsymbol{\sigma}^*$, and $\mathbf{p}^*$ that satisfy integral relation

$$\Phi_+ = \int_0^{t_f}\int_\Omega \varphi_+(\mathbf{u}, \boldsymbol{\sigma}, \mathbf{p})\, d\Omega\, dt = 0, \quad \varphi_+ = \frac{1}{2}[\rho \boldsymbol{\eta} \cdot \boldsymbol{\eta} + \boldsymbol{\xi} : \mathbf{C} : \boldsymbol{\xi}]; \tag{5}$$

$$\boldsymbol{\eta} = \mathbf{u}^{\bullet} - \rho^{-1}\mathbf{p}, \qquad \boldsymbol{\xi} = \boldsymbol{\varepsilon} - \mathbf{C}^{-1} : \boldsymbol{\sigma} \tag{6}$$

under equilibrium, kinematical, boundary, and initial conditions (2)–(4). Here the auxiliary velocity vector $\boldsymbol{\eta}$ and strain tensor $\boldsymbol{\xi}$ are introduced.

Note that the integrand φ_+ in Eq. (5) has the dimension of energy density and is nonnegative. Hence, the corresponding integral Φ_+ is nonnegative for arbitrary functions $\mathbf{u}$, $\boldsymbol{\sigma}$, and $\mathbf{p}$ ($\Phi_+ \geq 0$). The proposed integrodifferential problem (2)–(4) and (5) can be reduced to a variational one: to find unknown functions $\mathbf{u}^*$, $\boldsymbol{\sigma}^*$, and $\mathbf{p}^*$ minimizing the functional Φ_+

$$\Phi_+(\mathbf{u}^*, \boldsymbol{\sigma}^*, \mathbf{p}^*) = \min_{\mathbf{u}, \boldsymbol{\sigma}, \mathbf{p}} \Phi_+(\mathbf{u}, \boldsymbol{\sigma}, \mathbf{p}) = 0 \tag{7}$$

subject to constraints (2)–(4).

This variational statement for the initial-boundary value problem of controlled motions of an elastic body (1)–(4) is formulated with respect to the displacement vector $\mathbf{u}$, stress tensor $\boldsymbol{\sigma}$, and momentum density vector $\mathbf{p}$. After satisfying the equilibrium equation in Eq. (2) nine independent functions remain in the system. In numerical approaches to the dynamical problem this fact leads to sufficiently large dimension of system parameters. To decrease the number of unknown functions in the variational formulation and raise the effectiveness of numerical computation a special functional can be proposed

$$\Phi_0 = \int_0^{t_f} \int_\Omega \varphi_0(\mathbf{u}, \boldsymbol{\sigma}) d\Omega\, dt, \quad \varphi_0 = \frac{1}{2} \boldsymbol{\xi} : \mathbf{C} : \boldsymbol{\xi}. \tag{8}$$

To formulate a constrained minimization problem with Φ_0 the velocity-momentum vector relation $\boldsymbol{\eta} = 0$ in Eq. (6) must be considered as an additional differential constraint. After that the variational problem is to find unknown functions $\mathbf{u}^*$ and $\boldsymbol{\sigma}^*$ minimizing the functional Φ_0 under constraints

$$\begin{aligned}
&\Phi_0(\mathbf{u}^*, \boldsymbol{\sigma}^*) = \min_{\mathbf{u}, \boldsymbol{\sigma}} \Phi_0(\mathbf{u}, \boldsymbol{\sigma}) = 0, \\
&\rho \ddot{\mathbf{u}} = \nabla \cdot \boldsymbol{\sigma} + \mathbf{f}, \\
&\alpha_k(x) u_k + \beta_k(x) q_k = v_k, \quad x \in \gamma; \quad \mathbf{q} = \boldsymbol{\sigma} \cdot \mathbf{n}, \quad k = 1, 2, 3, \\
&\mathbf{u}(0, x) = \mathbf{u}^0(x), \quad \dot{\mathbf{u}}(0, x) = \rho^{-1} \mathbf{p}^0(x).
\end{aligned} \tag{9}$$

Denote the actual and arbitrary admissible displacements and stresses by $\mathbf{u}^*$, $\boldsymbol{\sigma}^*$ and $\mathbf{u}$, $\boldsymbol{\sigma}$, respectively, and specify that $\mathbf{u} = \mathbf{u}^* + \delta\mathbf{u}$, $\boldsymbol{\sigma} = \boldsymbol{\sigma}^* + \delta\boldsymbol{\sigma}$, then $\Phi_0(\mathbf{u}, \boldsymbol{\sigma}) = \delta_{\mathbf{u}}\Phi_0 + \delta_{\boldsymbol{\sigma}}\Phi_0 + \delta^2\Phi_0$. The second variation quadratic with respect to $\delta\mathbf{u}$, $\delta\boldsymbol{\sigma}$ is nonnegative ($\delta^2\Phi_0 \geq 0$). The fist variation of the functional Φ_0 is equal to zero for any admissible variations $\delta\mathbf{u}$, $\delta\boldsymbol{\sigma}$ if $\boldsymbol{\xi} = 0$.

The displacement vector $\mathbf{u}$ and stress tensor $\boldsymbol{\sigma}$ as well as their variations $\delta\mathbf{u}$ and $\delta\boldsymbol{\sigma}$ are related through the equilibrium equation shown in Eq. (9) and

$$\delta \ddot{\mathbf{u}} = \nabla \cdot \delta\boldsymbol{\sigma}. \tag{10}$$

Introduce the auxiliary tensor $\boldsymbol{\chi}$ so that

$$\mathbf{u} = \nabla \cdot \boldsymbol{\chi} + \int_0^t \int_0^t \mathbf{f}(\tau, x) d\tau d\tau, \quad \boldsymbol{\sigma} = \rho \ddot{\boldsymbol{\chi}} \tag{11}$$

and write down the stationary conditions for Φ_0

$$\begin{aligned}
&\delta_{\boldsymbol{\chi}}\Phi_0 = 0; \\
&\delta_{\boldsymbol{\chi}}\Phi_0 = \int_0^{t_f} \int_\Omega \left[\Delta_{\boldsymbol{\xi}} - \rho \ddot{\boldsymbol{\xi}}\right] : \delta\boldsymbol{\chi} d\Omega\, dt + \int_0^{t_f} \int_{\gamma_\sigma} \left[\left(\mathbf{n} \cdot \mathbf{C} : \boldsymbol{\xi}\right) \cdot \left(\nabla \cdot \delta\boldsymbol{\chi}\right) - \left(\nabla \cdot \mathbf{C} : \boldsymbol{\xi}\right) \cdot \left(\delta\boldsymbol{\chi} \cdot \mathbf{n}\right)\right] d\gamma dt + \\
&\qquad \iint_\Omega \left[\dot{\boldsymbol{\xi}} : \delta\boldsymbol{\chi} - \boldsymbol{\xi} : \delta\dot{\boldsymbol{\chi}}\right]_{t=0}^{t=t_f} d\Omega, \\
&\Delta_{\boldsymbol{\xi}} = \frac{1}{2}\left[\nabla\left(\nabla \cdot \mathbf{C} : \boldsymbol{\xi}\right) + \nabla\left(\nabla \cdot \mathbf{C} : \boldsymbol{\xi}\right)^{\mathrm{T}}\right].
\end{aligned} \tag{12}$$

The system of Euler's equations with the corresponding conditions on the boundary γ and at initial and terminal instants $t = 0, t_f$ can be obtained from Eq. (12). It is possible to show that the stationary conditions for the functional Φ_0 are equivalent to relations $\boldsymbol{\xi} = 0$ and together with constraint $\boldsymbol{\eta} = 0$ and Eqs. (2)–(4) constitute the full system of dynamical equations for linear elasticity.

4 Numerical algorithm and error analysis

Let us describe one of the possible algorithms approximating the solution $\mathbf{u}^*$, $\boldsymbol{\sigma}^*$ of the variational problem defined by Eq. (9). Constrain ourselves to the case of zero volume force $\mathbf{f}=0$. At the beginning the positive integers N_u and N_σ are chosen and the approximations $\tilde{\mathbf{u}}$ and $\tilde{\boldsymbol{\sigma}}$ of the solution are defined in the finite dimensional form

$$\tilde{\mathbf{u}} = \sum_{k=1}^{N_u} \mathbf{u}^{(k)} \psi_k(t,x), \quad \tilde{\boldsymbol{\sigma}} = \sum_{k=1}^{N_\sigma} \boldsymbol{\sigma}^{(k)} \psi_k(t,x). \tag{13}$$

Here $\{\psi_k(t,x), k=1,2,\ldots\}$ is some complete countable system of linearly independent functions and $\mathbf{u}^{(k)}$, $\boldsymbol{\sigma}^{(k)}$ are unknown real coefficients presented in the vector and tensor forms. The basis functions ψ_k should be chosen so that the approximations $\tilde{\mathbf{u}}$ and $\tilde{\boldsymbol{\sigma}}$ were able to satisfy exactly the equilibrium equation in Eq. (2) as well as boundary and initial conditions (3), (4). Therefore, the admissible set of the boundary vector $\mathbf{v}$ and initial functions $\mathbf{u}^0$ and $\mathbf{p}^0$ is settled. It is follows from Eq. (13) that the vectors $\mathbf{v}$, $\mathbf{u}^0$, $\mathbf{p}^0$ must have the structure

$$\mathbf{v} = \sum_{k=1}^{N_v} \mathbf{v}^{(k)} \psi_k\big|_{x\in\gamma}, \quad \mathbf{u}^0 = \sum_{k=1}^{N_u} \mathbf{u}^{(0k)} \psi_k(0,x), \quad \mathbf{p}^0 = \sum_{k=1}^{N_p} \mathbf{p}^{(0k)} \psi_k(0,x),$$

where $\mathbf{u}^{(0k)}$, and $\mathbf{p}^{(0k)}$ are fixed coefficients; $\mathbf{v}^{(k)}$ are given constants on the boundary γ.

In the next step the conditions (2)–(4) are satisfied with respect to unknown coefficients $\mathbf{u}^{(k)}$, $\boldsymbol{\sigma}^{(k)}$ and the admissible approximations $\tilde{\mathbf{u}}$, $\tilde{\boldsymbol{\sigma}}$ obtained are substituted into the functional Φ_0. Since the functional Φ_0 is quadratic with respect to the parameters $\mathbf{u}^{(k)}$, $\boldsymbol{\sigma}^{(k)}$, the minimization problem (9) is reduced to the linear system of the algebraic equations versus unknown coefficients $\mathbf{u}^{(k)}$, $\boldsymbol{\sigma}^{(k)}$.

To estimate the quality of the numerical solution $\tilde{\mathbf{u}}(t,x)$ and $\tilde{\boldsymbol{\sigma}}(t,x)$ the following criterion is proposed [18]

$$\begin{aligned}
&\Delta = \tilde{\Phi}_0 / \tilde{\Psi} < \delta, \quad \tilde{\Phi}_0 = \Phi_0(\tilde{\mathbf{u}}, \tilde{\boldsymbol{\sigma}}); \\
&\tilde{\Psi} = \int_0^{t_f} \tilde{W}\, dt, \quad \tilde{W}(t) = \int_\Omega \psi(\tilde{\mathbf{u}}) d\Omega, \quad \psi = \frac{1}{2}\left[\rho \mathbf{u}^{\boldsymbol{\cdot}} \cdot \mathbf{u}^{\boldsymbol{\cdot}} + \boldsymbol{\varepsilon} : \mathbf{C} : \boldsymbol{\varepsilon}\right] \\
&\tilde{\varphi}_0(t,x) = \frac{1}{2}\left[\tilde{\boldsymbol{\xi}} : \mathbf{C} : \tilde{\boldsymbol{\xi}}\right], \quad \tilde{\boldsymbol{\xi}} = \tilde{\boldsymbol{\varepsilon}} - \mathbf{C}^{-1} : \tilde{\boldsymbol{\sigma}} \\
&\dot{\tilde{W}} = \int_\gamma \tilde{\mathbf{q}} \cdot \tilde{\mathbf{u}}^{\boldsymbol{\cdot}} d\gamma + \dot{W}_{err}, \quad \dot{W}_{err} = \int_\Omega \left[\tilde{\boldsymbol{\xi}} : \mathbf{C} : \tilde{\boldsymbol{\varepsilon}}^{\boldsymbol{\cdot}}\right] d\Omega, \quad \tilde{\mathbf{q}} = \mathbf{C} : \tilde{\boldsymbol{\sigma}}.
\end{aligned} \tag{14}$$

Here δ is a small positive constant, $\tilde{\Phi}_0 \geq 0$ is the value of functional Φ_0 on the numerical solution, $\Psi > 0$ is the time integral of the total mechanical energy W, and ψ is the volume density of this energy. The ratio Δ can serve as a relative integral error of the approximate solution $\tilde{\mathbf{u}}$ and $\tilde{\boldsymbol{\sigma}}$, whereas function $\tilde{\varphi}_0$ shows the distribution of local error. The time derivative (the power) $\dot{\tilde{W}}$ includes a term $\dot{W}_{err}$ which shows the energy change rate caused by system discretization. It is follows from Eq. (14) that the value of parasitic energy $\int_0^{t_f} \left|\dot{W}_{err}\right| dt$ is related to the value of the error Δ.

5 3D beam lateral motions

As an example of algorithm implementation, let us consider the 3D problem of lateral controlled motions for the rectilinear beam with a quadratic cross section (see Fig. 1). The sizes of the cross section do not change along the beam length. It is supposed that the static volume forces are absent ($\mathbf{f}=0$) and the beam is made of homogeneous and isotropic material with given Young's modulus E, Poisson's ratio μ, and volume density ρ. The geometrical beam parameters such as the beam length L and structural height $2a$ as well as terminal time are fixed, and hence the problem is defined in the following 4D time-space domain

$$\Sigma = \{t, x : t \in (0,T), x \in \Omega\}, \quad \Omega = \left\{x_1, x_2, x_3 : 0 < x_1 < L, |x_2| < a, |x_3| < a\right\}.$$

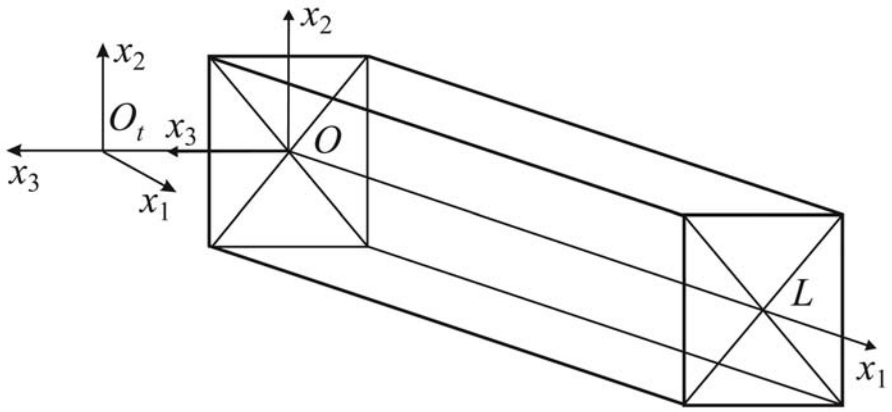

Figure 1. Rectilinear elastic beam

Let us introduce the fixed reference frame $Ox_1x_2x_3$. Its origin O coincides with the central point of the left beam cross section in the initial time. The x_1-axis is directed along the beam and the coordinate axes Ox_2 and Ox_3 are parallel to the cross section sides. The boundary conditions under consideration are

$$\begin{aligned}&\sigma_{12}\big|_{x_2=\pm a}=\sigma_{22}\big|_{x_2=\pm a}=\sigma_{23}\big|_{x_2=\pm a}=\sigma_{13}\big|_{x_3=\pm a}=\sigma_{23}\big|_{x_3=\pm a}=\sigma_{33}\big|_{x_3=\pm a}=0\\&\sigma_{11}\big|_{x_1=L}=\sigma_{12}\big|_{x_1=L}=\sigma_{13}\big|_{x_1=L}=0,\quad u_1\big|_{x_1=0}=u_2\big|_{x_1=0}=0\end{aligned}\tag{15}$$

The control boundary function v is the time-dependent polynomial displacement u_3 of the left beam cross section:

$$u_3\big|_{x_1=0}=v=\sum_{k=2}^{N_v}v_k t^k.\tag{16}$$

Suppose also that in the initial instant $t=0$ the elastic displacements of the beam are absent and all its points are at rest. So in Eqs. (4)

$$\mathbf{u}^0(x)=\mathbf{p}^0(x)=0.\tag{17}$$

To find an approximate solution and optimize control in the problem of 3D longitudinal beam dynamics a polynomial representation of the unknown functions was used by Kostin and Saurin [16]. In Kostin and Saurin [12] bivariate piece-wise polynomial splines defined on rectangular meshes were applied to modeling and optimization of lateral Bernoulli beam motions. In this example a finite element approach and spline techniques are used to model 3D beam dynamics.

Let us consider the variational problem (9), fix an approximation order M_p, and choose the following approximations $\tilde{u}_k^{(M_p)}$, $\tilde{\sigma}_{kl}^{(M_p)}$, $k,l=1,2,3$, for unknown components of displacement vector $\mathbf{u}$ and the stress tensor $\boldsymbol{\sigma}$

$$\begin{aligned}&\tilde{\sigma}_{11}^{(M_p)}=\sum_{i+j=0}^{M_p}\sigma_{11}^{(i,j)}(t,x_1)x_2^{2i}x_3^{2j+1},\quad \tilde{\sigma}_{23}^{(M_p)}=\sum_{i+j=0}^{M_p-2}\sigma_{23}^{(i,j)}(t,x_1)(a^2-x_2^2)(a^2-x_3^2)x_2^{2i-1}x_3^{2j-2},\\&\tilde{\sigma}_{22}^{(M_p)}=\sum_{i+j=0}^{M_p}\sigma_{22}^{(i,j)}(t,x_1)(a^2-x_2^2)x_2^{2i}x_3^{2j+1},\quad \tilde{\sigma}_{33}^{(M_p)}=\sum_{i+j=0}^{M_p}\sigma_{33}^{(i,j)}(t,x_1)(a^2-x_3^2)x_2^{2i}x_3^{2j+1},\\&\tilde{\sigma}_{12}^{(M_p)}=\sum_{i+j=0}^{M_p-1}\sigma_{12}^{(i,j)}(t,x_1)(a^2-x_2^2)x_2^{2i+1}x_3^{2j+1},\quad \tilde{\sigma}_{13}^{(M_p)}=\sum_{i+j=0}^{M_p}\sigma_{13}^{(i,j)}(t,x_1)(a^2-x_3^2)x_2^{2i}x_3^{2j};\\&\tilde{u}_1^{(M_p)}=\sum_{i+j=0}^{M_p}u_1^{(i,j)}(t,x_1)x_2^{2i}x_3^{2j+1},\quad \tilde{u}_2^{(M_p)}=\sum_{i+j=0}^{M_p-1}u_2^{(i,j)}(t,x_1)x_2^{2i+1}x_3^{2j+1},\quad \tilde{u}_3^{(M_p)}=\sum_{i+j=0}^{M_p}u_3^{(i,j)}(t,x_1)x_2^{2i}x_3^{2j}.\end{aligned}\tag{18}$$

Here $u_k^{(i,j)}$ and $\sigma_{kl}^{(i,j)}$ are functions of the time t and coordinate x_1 which will be defined below. These approximations obey the boundary conditions on the beam sides parallel to the x_1-axis.

The symmetry with respect to coordinate planes Ox_1x_2 and Ox_1x_3 allows one to decompose the original problems to four independent subproblems including 3D beam compression-stretching, bending around Ox_2 axis, bending around Ox_3 axis, and torsion. In this example the control displacement in Eq. (16) excites only bending motions around Ox_2 axis. The polynomials with respect to the variables x_2 and x_3 proposed in Eq. (18) do not violate the symmetry properties of problem (9) under the boundary and initial conditions (15)–(17) as it has been shown in [20].

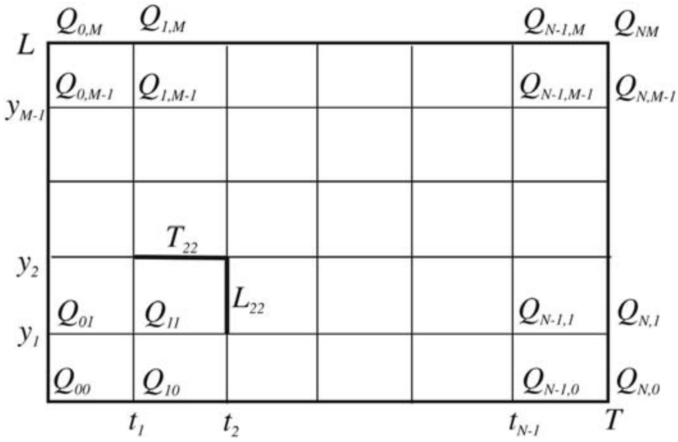

Figure 2. Space-time mesh.

Divide the two-dimensional time-space domain $\{t,x_1\}\in\Upsilon=(0,T)\times(0,L)$ on $N\times M$ rectangles Υ_{kl} which vertices have coordinates $Q_{k-1,l-1}$, $Q_{k-1,l}$, $Q_{k,l-1}$, $Q_{k,l}$, where $Q_{k,l}=\{t_k,y_l\}$; $t_k>t_{k-1}$, $k=1,\ldots,N$; $y_l>y_{l-1}$, $l=1,\ldots,M$; $t_0=0$, $t_N=T$, $y_0=0$, $y_M=L$ (see Fig. 2). Let also the boundary edges of these time-space rectangles be named $L_{kl}=(Q_{k,l-1},Q_{kl})$, $k=0,\ldots,N$, $l=1,\ldots,M$, and $T_{kl}=(Q_{k-1,l},Q_{kl})$, $k=1,\ldots,N$, $l=0,\ldots,M$. In each 4D time-space subdomain

$$\Sigma_{kl}=\left\{t,x_1,x_2,x_3:(t,x_1)\in\Upsilon_{kl},|x_2|<a,|x_3|<a\right\}.$$

approximating polynomials $u_\alpha^{(i,j)}$ and $\sigma_{\alpha\beta}^{(i,j)}$ are given

$$\begin{aligned}
&u_\alpha^{(j_2,j_3)}(t,x_1)=\sum_{j_0+j_1=0}^{N_p}u_\alpha^{(J)}t^{j_0}x_1^{j_1},\quad \sigma_{\alpha\beta}^{(j_2,j_3)}(t,x_1)=\sum_{j_0+j_1=0}^{N_p}\sigma_{\alpha\beta}^{(J)}t^{j_0}x_1^{j_1},\quad \alpha,\beta=1,2,3;\\
&J=\{j_i\},\quad i=0,\ldots,5;\quad j_0,j_1\le N_p,\quad j_2,j_3\le M_p,\quad j_4=0,\ldots,N,\quad j_5=0,\ldots,M.
\end{aligned}\tag{19}$$

Here $u_\alpha^{(J)}$ and $\sigma_{\alpha\beta}^{(j_2,j_3)}$ are unknown real coefficients. The integer N_p is chosen so that the equilibrium equation and zero initial conditions in Eq. (9), boundary conditions (15), (16) can be exactly satisfied. In addition, to apply the variational formulation given above the following conformed interelement relations must obey

$$\begin{aligned}
&\{t,x_1\}\in L_{kl},\quad k=1,\ldots,N-1,\quad l=1,\ldots,M:\\
&\quad\tilde{\mathbf{u}}^{(kl)}(t_k,x_1,x_2,x_3)=\tilde{\mathbf{u}}^{(k+1,l)}(t_k,x_1,x_2,x_3),\quad \tilde{\mathbf{u}}^{\cdot(kl)}(t_k,x_1,x_2,x_3)=\tilde{\mathbf{u}}^{\cdot(k+1,l)}(t_k,x_1,x_2,x_3);\\
&\{t,x_1\}\in T_{kl},\quad k=1,\ldots,N,\quad l=1,\ldots,M-1:\\
&\quad\tilde{\mathbf{u}}^{(kl)}(t,y_l,x_2,x_3)=\tilde{\mathbf{u}}^{(k,l+1)}(t,y_l,x_2,x_3),\\
&\quad\mathbf{n}\cdot\tilde{\boldsymbol{\sigma}}^{(kl)}(t,y_l,x_2,x_3)=\mathbf{n}\cdot\tilde{\boldsymbol{\sigma}}^{(k,l+1)}(t,y_l,x_2,x_3),\quad \mathbf{n}=(1,0,0).
\end{aligned}\tag{20}$$

After satisfying local constraints in Eq. (9) and interelement conditions (20) the resulted finite-dimensional unconstrained minimization problem yields an approximate solution $\tilde{\mathbf{u}}^*(t,x,v)$, $\tilde{\boldsymbol{\sigma}}^*(t,x,v)$ for arbitrary control parameters v_k in Eq. (16).

6 Numerical results

In this section the numerical results of modeling the 3D lateral beam motion described in Section 5 are presented. The following dimensionless geometrical and mechanical parameters are chosen:

$$L=1,\quad a=0.05,\quad T=2,\quad \rho S=1,\quad EI=1,\quad \nu=0.3,\quad S=4a^2,\quad I=4a^4/3,$$

and the mesh and approximation numbers are assigned: $N=5$, $M=1$, $M_p=2$, $N_p=10$. For numerical solution of this dynamical problem a time uniform mesh with node instants $t_i=Ti/N$, $i=1,\ldots,N$, is used. After satisfying equilibrium equation, initial, boundary, and interelement conditions, the total number of degrees of freedom is equal to $N_{DOF}=1985$. The following function of lateral displacement for the left beam cross section is fixed as a sample control law

$$v = (3t^2 - t^3)/4, \quad v(0) = \dot{v}(0) = \dot{v}(T) = 0, \quad v(T) = 1. \tag{21}$$

For the given system data the estimated value of the energy time integral is $\Psi = 0.1822$. The absolute and relative integral error defined in Eq. (14) are $\Phi_0 = 0.0020$, $\Delta_0 = 1.1\%$.

In the Fig. 3 the relative displacement of the central point $u_3(t, L, 0, 0) - v(t)$ of the right free beam cross section versus time t is shown.

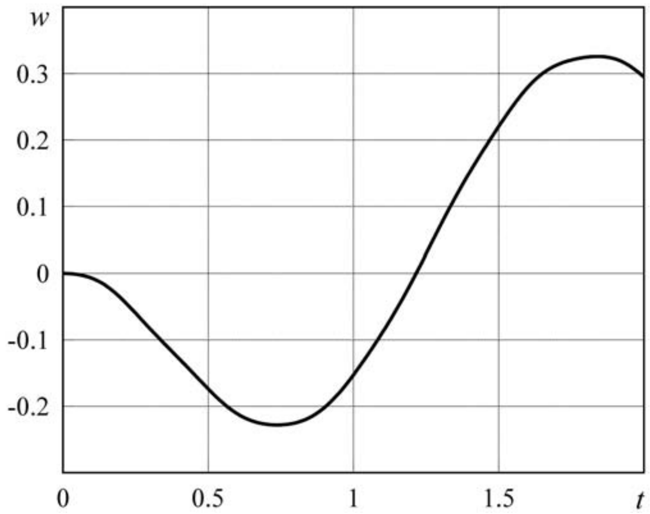

Figure 3. Relative displacement at the central point of the right free beam cross section vs. time.

In Fig. 4 the function of energy linear density $\psi_1(t, x_1) = \int_{-a}^{a}\int_{-a}^{a} \psi \, dx_2 dx_3$ is depictured. The distribution of solution local error $\varphi_1(t, x_1) = \int_{-a}^{a}\int_{-a}^{a} \varphi_0 dx_2 dx_3$ along axis Ox_1 is presented in Fig. 5. As it is seen from the figure the maximal errors is concentrated in the area near the left cross section of the beam.

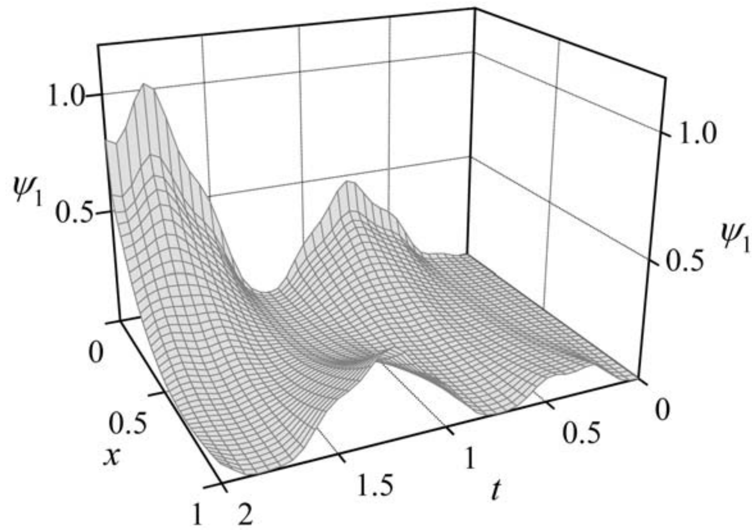

Figure 4. Linear density of the total mechanical energy along beam axis.

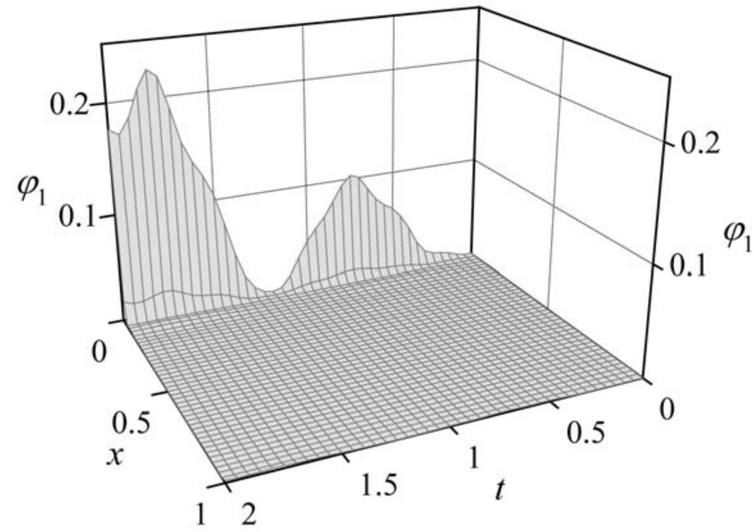

Figure 5. Distribution of solution local error along beam axis.

The 3D beam model takes into account space deflections in any beam cross section at an arbitrary control instant. As an example, the deformed cross-section shape of the beam at $t = 0.8$ and $x_1 = 0.05$ is displayed in Fig. 6. The local error distribution φ_0 in this cross section at the same time is shown in Fig. 7. The error function on this rectangle reaches its maximal values at the beam edges

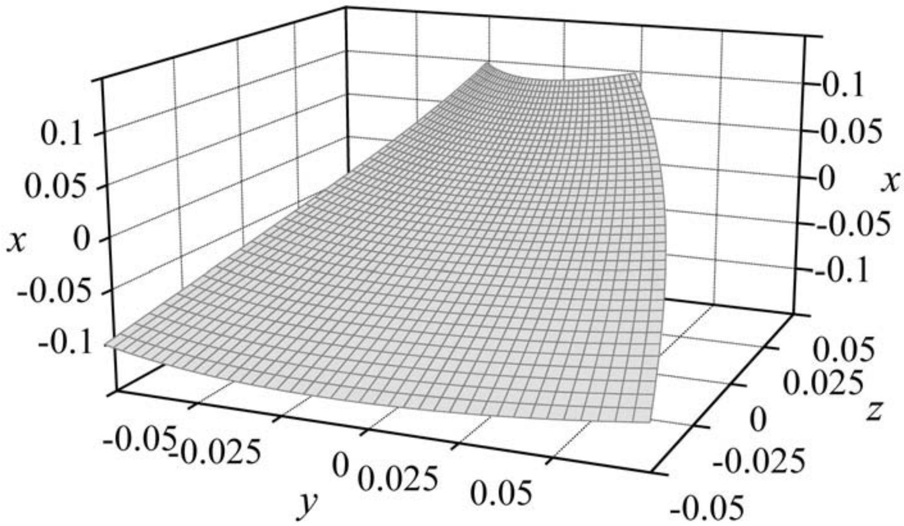

Figure 6. Deformed cross-section shape.

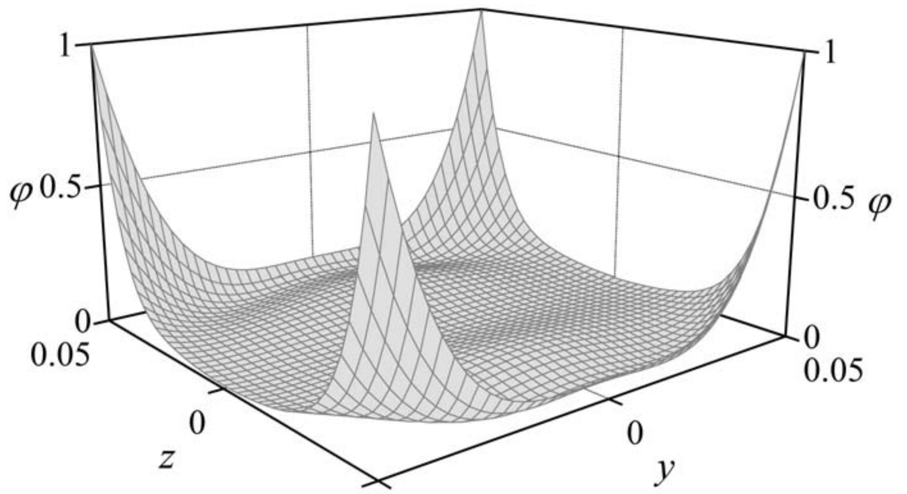

Figure 7. Distribution of local error in a beam cross section at a fixed time instant.

In this control process the total mechanical energy W starting with zero value reaches its maximum during this motion as depictured in Fig. 8. The energy change rate $\dot{W}_{err}$ caused by system discretization is reflected in Fig. 9. The numerical parasitic power defined in Eq. (14) result in noticeable energy underestimate. The linear density distribution along beam axis Ox_1 $\omega(t,x_1) = \int_{-a}^{a}\int_{-a}^{a} \tilde{\xi} : \mathbf{C} : \tilde{\varepsilon}^{\bullet} dx_2 dx_3$ for these numerical disbalance is shown in Fig. 10. As it is seen from the figure the maximal energy parasitic source is, analogously to the error distribution φ_1, the area near the left cross section of the beam.

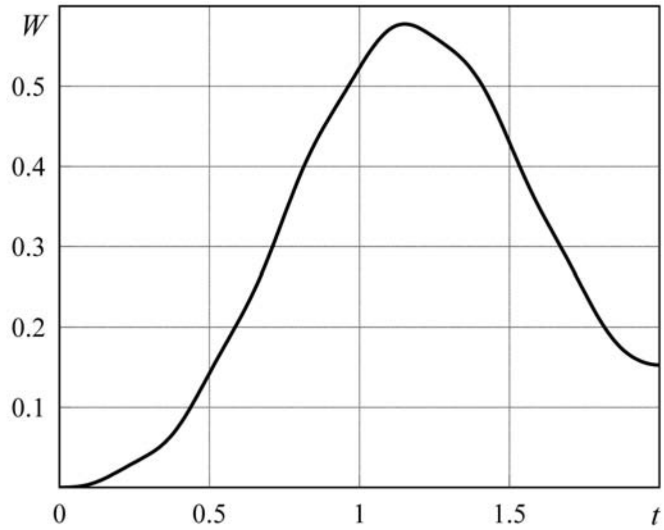

Figure 8. Mechanical energy versus time.

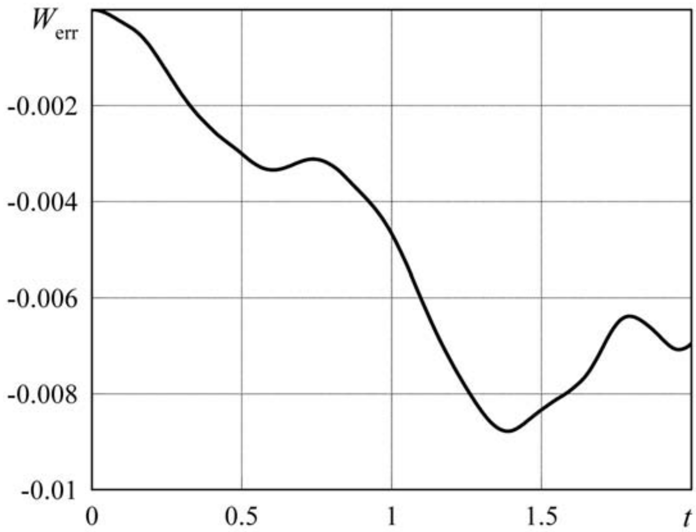

Figure 9. Energy changes over numerical parasitic disbalance.

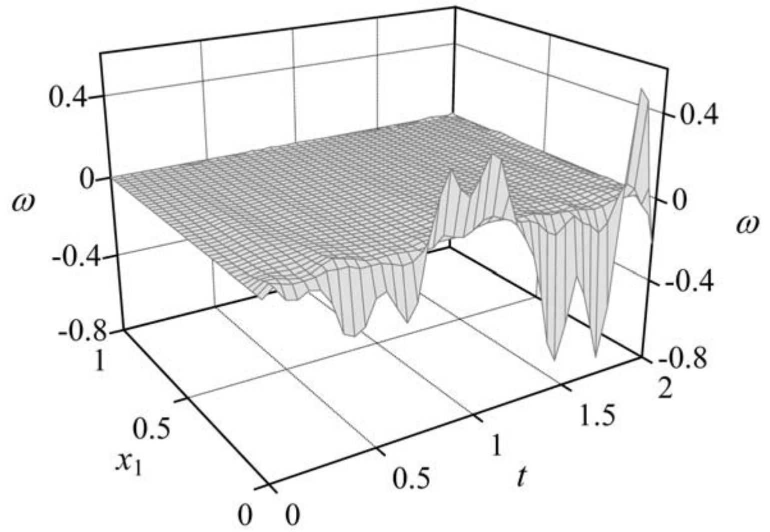

Figure 10. Linear density distribution along beam axis for the numerical energy disbalance.

7 Conclusions

Based on the method of integrodifferential relations a new variational principle which stationary conditions are equivalent to the constitutive relation was deduced for the initial boundary value problems of linear elasticity. For this principle the nonnegative functional under minimization can serve as integral criteria of the solution quality, whereas its integrand characterizes the local error distribution. The effective numerical algorithm used the time-space piece-wise polynomial approximations enables one to construct effective estimates for various integral characteristics (elastic energy, displacements, etc.). This algorithm can be directly applied to nonhomogeneous anisotropic structures. The case of more complex boundary conditions such as aero- and hydrodynamic forces, nonconservative loading, etc. does not encounter principal difficulties. The FEM realization will give one the possibility to work out various strategies of p-h adaptive mesh refinement by using a local error estimate. The method can appear to be useful also in advanced beam, plate, and shell theories. The approach worked out can be also applied to other inverse mechanical problems such as shape optimization, identification, and so on as well as to optimal control problems with inequality constraints.

8 Acknowledgments

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