

# NUMERICS OF THE VAN-DER-POL EQUATION WITH RANDOM PARAMETER

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**Abstract.** In this article, the problem of long-term integration in the Wiener-Hermite calculus for ordinary differential equations with random parameters is discussed. The Wiener-Hermite calculus is known to lose accuracy with increasing time, even if it provides good results for short times. This is demonstrated for the Van-der-Pol equation with a random parameter. To reduce this problem, the adapted stochastic spectral method (aSSM), which is based on the work of P. Vos [8], is presented. Applying aSSM to the Van-der-Pol equation reduces the error considerably and allows an accurate long-term integration.

## 1 Introduction

This paper deals with the method of stochastic spectral expansion of random variables with finite variance. The stochastic spectral expansion is based on the work of N. Wiener [11] and was considered in the framework of generalized Fourier series by R. Ghanem and P. S. Spanos, see [2]. Karniadakis, LeMaitre and co-workers developed the method [14, 6, 5], which is still an active field of research [1].

Problems arising in industry possess large dimensions and are time consuming to compute, even in the deterministic case. Therefore in the case of uncertain data, they are often not treatable with standard techniques like Monte Carlo simulation. The method of stochastic spectral expansion, or one of its derivatives, often provides a good means to handle such problems. We choose the Van-der-Pol equation as benchmark problem, following P. E. Zadunaisky, who stated "I have a theory that whenever you want to get in trouble with a method, look for the Van-der-Pol equation.", see [3].

Although its simplicity in application, the calculus of stochastic spectral expansion does not provide a general purpose method. For example the costs increase exponentially with the number of random parameters in the underlying model. Furthermore, no mechanisms for error estimation and control are available. Up to now, the standard way to check accuracy is to compare the results with a reference solution computed by Monte-Carlo simulation. This is not satisfactory in practical use.

To reduce this problem, we developed the adapted stochastic spectral method (aSSM). By adapting the density while computing the solution, we are able to reduce the error considerably in contrast to the standard stochastic spectral decomposition. The work is organized as follows. At first the Van-der-Pol equation is derived from modeling the electrical circuit shown in figure 1. Subsequently, the application of the stochastic spectral decomposition method to the randomness dependent Van-der-Pol equation is discussed. After that, the algorithm of aSSM is elaborated and numerical results of its application to the random Van-der-Pol equation are presented. We close this work by stating conclusions and objectives for future work.

## 2 Derivation of the Van-der-Pol equation

Preliminarily we derive the Van-der-Pol equation (VDP) from the modeling of the electrical circuit shown in figure 1. It consists of an inductance  $L$ , a capacitor  $C$ , a tunnel diode and the grounding  $U_0$  as well as an operational voltage  $U_{op}$ . We are interested in the voltage at the nodes  $N_1$  and  $N_2$ . Modeling electrical circuits is based on the two fundamental rules of Kichhoff, which can be stated as follows:

1. the sum of all currents in each node equals zero,
2. the sum of all partial voltages in a loop equals zero.

Furthermore, we need the model equations for the basic components of the electrical circuit [4]

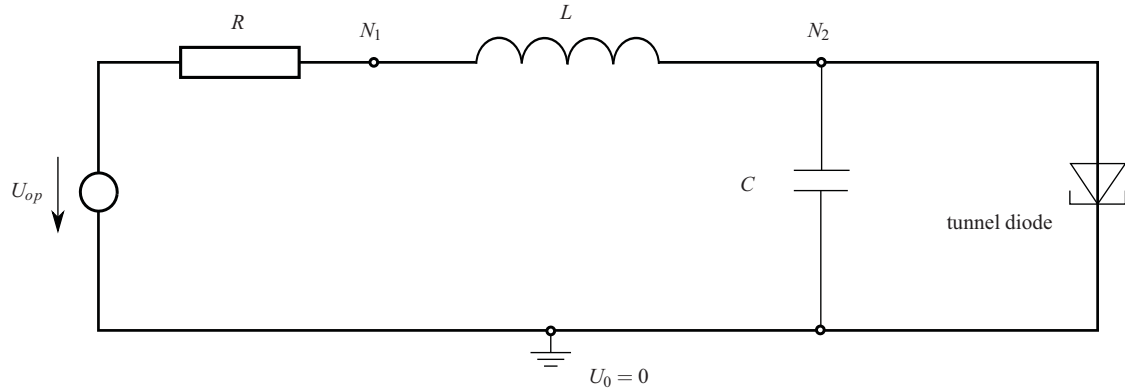
$$U_L = L\dot{I}_L, \tag{1}$$

$$I_C = C\dot{U}_C, \tag{2}$$

$$U_R = RI_R, \tag{3}$$

$$I_D = f(U_D). \tag{4}$$

Here,  $U_*$  and  $I_*$  denote the respective voltages and currents for  $L$  the inductance,  $C$  the capacitor,  $R$  the resistance and  $D$  the tunnel diode. The function  $f$  describes the characteristics for the tunnel diode. We use the notation  $\dot{\cdot} := d/dt$ .



**Figure 1:** Circuit with an inductance  $L$ , a capacitor  $C$ , a resistance  $R$ , a tunnel diode and the grounding  $U_0$  as well as an operational voltage  $U_{op}$ . Additionally two nodes  $N_1$  and  $N_2$  are plotted.

Having settled the basic physical laws, we develop the mathematical model for the circuit in figure 1. We denote the voltage at node  $N_i$  by  $U_i$ ,  $i = 1, 2$ . The current  $I_{1 \rightarrow op}$  between  $N_1$  and the operational voltage, the current  $I_{2 \rightarrow C \rightarrow 0}$  from node  $N_2$  to the ground via the capacitor and the current  $I_{2 \rightarrow D \rightarrow 0}$  from node  $N_2$  to the ground via the tunnel diode are given by

$$\begin{aligned} I_{1 \rightarrow op} &= (U_1 - U_{op})/R, \\ I_{2 \rightarrow C \rightarrow 0} &= C(\dot{U}_2 - \dot{U}_0), \\ I_{2 \rightarrow D \rightarrow 0} &= f(U_2 - U_0), \end{aligned} \tag{5}$$

which are direct applications of (2)-(4). Using Kirchhoff's rules and equations (5) we get

$$0 = \frac{1}{R}(U_1 - U_{op}) + I_{1 \rightarrow 2}, \tag{6}$$

$$0 = I_{2 \rightarrow 1} + C(\dot{U}_2 - \dot{U}_0) + f(U_2 - U_0). \tag{7}$$

The currents  $I_{1 \rightarrow 2}$  and  $I_{2 \rightarrow 1}$  are defined by the same systematics as indicated above, denoting the currents from node  $N_1$  to  $N_2$  and reverse. Differentiation of (6) with respect to time yields

$$0 = \frac{1}{R}(\dot{U}_1 - \dot{U}_{op}) + \frac{1}{L}(U_1 - U_2), \tag{8}$$

$$0 = \frac{1}{R}(U_1 - U_{op}) + C\dot{U}_2 + f(U_2), \tag{9}$$

where we already introduced (1) for  $\dot{I}_{1 \rightarrow 2}$ , exploited the fact that  $I_{2 \rightarrow 1} = -I_{1 \rightarrow 2}$  and set  $U_0 \equiv 0$ . The combination of equations (8) and (9) results in

$$0 = C\ddot{U} + \left( \frac{RC}{L} + \frac{d}{dU}f(U) \right) \dot{U} + \frac{1}{L}(Rf(U) + U - U_{op}), \tag{10}$$

omitting the subscript of  $U_2$ . We fit a polynomial of degree 3 to the characteristic curve of the diode

$$f(U) = a_1U + a_2U^2 + a_3U^3,$$

so that the well known nonlinear Van-der-Pol oscillator results from equation (10)

$$0 = \ddot{y}(t) + p(y(t)^2 - 1)\dot{y}(t) + y(t). \tag{11}$$

The parameter  $p$  represents a combination of the polynomial coefficients  $a_i$ ,  $i = 1, 2, 3$ , and denotes the uncertainty in the following.

	distribution	orthogonal polynomial	support
continuous	Gaussian	Hermite	$(-\infty, \infty)$
	Gamma	Laguerre	$[0, \infty)$
	Beta	Jacobi	$[a, b]$
	Uniform	Legendre	$[a, b]$
discrete	Poisson	Charlier	$\{0, 1, 2, \dots\}$
	Binomial	Krawtchouk	$\{0, 1, \dots, n\}$
	Negative binomial	Meixner	$\{0, 1, 2, \dots\}$
	Hypergeometric	Hahn	$\{0, 1, \dots, n\}$

**Table 1:** Distributions and the corresponding orthogonal polynomials ( $n \in \mathbb{N}$  and  $a, b \in \mathbb{R}$ ).

### 3 Stochastic spectral expansion and stochastic domain decomposition

In this section we introduce the stochastic spectral expansion, also known as Wiener-Hermite expansion, which is an expansion of random variables with finite variance into a series of Hermite polynomial functionals. It is based on the pioneering work [11] of N. Wiener in 1938. In [2] R. Ghanem and P. D. Spanos put it into the framework of generalized Fourier series, in order to work more easily. Further advancements were introduced in [14] by D. Xiu and G. E. Karniadakis. They proposed the use of arbitrary basis functionals of  $L_2(P)$  with respect to an appropriate choice of weight function  $\rho$ . Assume  $X(\xi) \in L_2(P)$  is a random variable and  $\rho$  is the density function, which we assume to exist, of the probability measure  $P$ . Further we denote the respective orthogonal polynomial functionals by  $\{\Psi_i\}_{i=0}^\infty$ . If they belong to the Askey scheme of orthogonal polynomial functionals, the generalized Wiener-Hermite expansion

$$X(\xi) = \sum_{i=0}^\infty q_i \Psi_i(\xi)$$

is called Wiener-Askey expansion of  $X$ . It is a remarkable fact, that the density functions for all common distributions correspond to weight functions of orthogonal polynomial functionals in the Askey scheme, see table 1.

But not only orthogonal polynomial functionals from the Askey scheme are suitable for the stochastic spectral expansion. Every complete basis of  $L_2(P)$  can be chosen. In [6, 5] O. LeMaître et. al. used Wavelets as basis functionals. In this article, we concentrate on orthogonal polynomial functionals, which are orthogonal with respect to arbitrary weight functions.

A further advancement of the Wiener-Hermite expansion is the stochastic domain decomposition. That means, that the underlying probability space is divided into elements. This method was proposed by G. E. Karniadakis et. al. in [9], where it is called Multi Elements generalized Polynomial Chaos (MEgPC). For the reason, that it will be supplementary used in the adaptive Stochastic Spectral Method, we give a short introduction on stochastic domain decomposition. For simplicity we restrict to the one dimensional case.

Let  $y(\xi) : S \rightarrow I$  be a random variable on the probability space  $(S, \mathcal{F}(S), P)$  onto an interval  $I \subset \mathbb{R}$  and assume a decomposition of  $S$  into  $n$  disjunct intervals  $\{S_k\}_{k=1}^n$  is given. We denote the density of  $y$  by  $\rho : I \rightarrow \mathbb{R}_0^+$ . The decomposition of  $S$  into disjunct intervals will be prescribed by aSSM as we will see in section 5.2. The stochastic spectral expansion of  $y$  restricted to  $S_k$ , denoted by  $y_k$ , can be computed as follows: at first we compute the conditional density  $\rho_k$  of  $y_k$ ,  $k = 1, \dots, n$ , by

$$\rho_k(\xi) = \frac{\rho(\xi)}{P(S_k)}.$$

Secondly, orthogonal polynomial functionals  $\{\Psi_{i,k}\}_{i=0}^\infty$  with respect to  $\rho_k$  are constructed on  $S_k$ , using a stable Gram-Schmidt orthogonalization procedure. This procedure results in a number of  $n$  stochastic spectral expansions

$$y_k(\xi) = \sum_{i=0}^\infty q_{i,k} \Psi_{i,k}(\xi), \quad \text{for } k \in \{1, \dots, n\}.$$

Using these expansions, we are able to compute the moments of  $y$ . For example the expected value is computed by

$$\mathbb{E}(y) = \int_S y(\xi) \rho(\xi) d\xi = \sum_{k=1}^n P(S_k) \int_{S_k} y_k(\xi) \rho_k(\xi) d\xi = \sum_{k=1}^n P(S_k) q_{0,k}.$$

Analogously, the variance can be computed by

$$\mathbb{V}(y) = \int_S y(\xi)^2 \rho(\xi) d\xi - \mathbb{E}(y)^2 = \sum_{k=1}^n P(S_k) \sum_{i=0}^\infty q_{i,k}^2 - \mathbb{E}(y)^2,$$

where we used Parseval's equality, see f.e. [10]. The variance will be approximated by

$$\mathbb{V}(y) \approx \sum_{k=1}^n P(S_k) \sum_{i=0}^m q_{i,k}^2 - \mathbb{E}(y)^2,$$

and  $m \in \mathbb{N}$  is called the order of the stochastic spectral expansion. In all the calculations above we assumed the polynomial functionals to be normalized to have the norm equal to 1.

### 4 The stochastic Van-der-Pol equation

We now return to the Van-Der-Pol equation (11) and assume the parameter  $p$  to be a random variable. We choose the range measure to be  $P_0 := \mathcal{U}[1.9, 2.1]$ . Its density is given by

$$\rho^{(0)}(p) = \begin{cases} 5 & \text{if } p \in [1.9, 2.1] \\ 0 & \text{else.} \end{cases}$$

The solution becomes a stochastic process  $y(t, p)$  and the Van-der-Pol equation becomes a random differential equation

$$0 = \ddot{y}(t, p) + p(y(t, p)^2 - 1)\dot{y}(t, p) + y(t, p). \tag{12}$$

We assume the initial conditions to be prescribed deterministically by  $y(0, p) \equiv 0.1$  and  $\dot{y}(0, p) \equiv 0.0$ . The density of  $y(t_i, p)$  for a fixed time  $t_i$  will be denoted by  $\rho^{(i)}$ . There are two possible ways to apply the Wiener-Askey calculus to the random Van-der-Pol equation. The first one works as follows: we insert the truncated expansion

$$y(t, p) \approx \sum_{i=0}^m q_i(t)\Psi_i(p)$$

into (12)

$$0 = \sum_{i=0}^m \ddot{q}_i(t)\Psi_i(p) + p \left( \left( \sum_{i=0}^m q_i(t)\Psi_i(p) \right)^2 - 1 \right) \sum_{i=0}^m \dot{q}_i(t)\Psi_i(p) + \sum_{i=0}^m q_i(t)\Psi_i(p),$$

and project onto the finite dimensional subspace  $F := \text{span}\{\Psi_0, \dots, \Psi_m\} \subset L_2(P_0)$ , so that we arrive at an  $m$ -dimensional system of deterministic ordinary differential equations in the unknown coefficient functions  $\{q_i\}_{i=0}^m$ . These fully coupled differential equations can be solved by using suitable integrators.

The second way of applying the Wiener-Askey calculus is called the non-intrusive method, or stochastic collocation, [12]. Here we use the properties of the Fourier series and compute the coefficient functions directly by integration:

$$q_i(t) = \int_S y(t, p)\Psi_i(p)\rho(p)dp, \tag{13}$$

for  $i = 1, \dots, m$ . Although the following method of adapted stochastic spectral expansions can be applied to both methods of application, we will restrict to the non-intrusive method.

We now discuss the problem arising from the long-term integration of random differential equations. Therefore we consider the Wiener-Askey expansion

$$y(t, p) = \sum_{i=0}^{\infty} q_i(t)\Psi_i(p),$$

and compute the coefficient functions  $\{q_i\}_{i=0}^{\infty}$  using formula (13). The polynomial functionals  $\{\Psi_i\}_{i=0}^{\infty}$  are chosen, so that they are orthogonal with respect to the weight function  $\rho^{(0)}$ , which is the density of the random parameter  $p$ . This choice is reasonable, if we consider the mapping

$$T(t, p) : \begin{cases} L_2(\Omega, \mathcal{F}, P) & \rightarrow L_2(\Omega, \mathcal{F}, P) \\ p & \mapsto y(t, p) \end{cases}$$

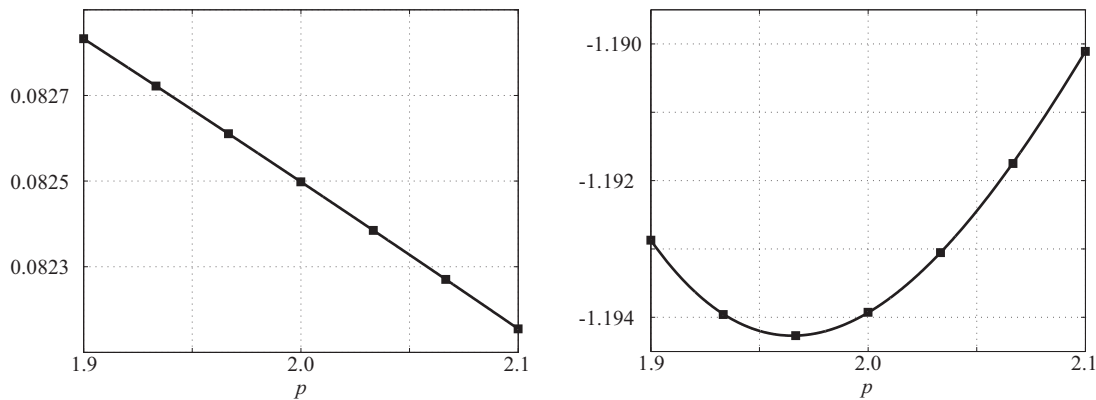
being well approximated by a linear mapping for early time  $t > 0$ . In this case, the distribution of  $y(t, \cdot)$  is approximately a linear transformation of  $P_0$  and the Wiener-Askey expansion of  $y(t, \cdot)$  is supposed to converge very fast. If  $t$  becomes large, the influence of the in general non-linear character of  $T$  becomes prominent and  $P_0$  becomes a bad approximation of the distribution of  $y(t, \cdot)$ . Hence, convergence of the Wiener-Askey expansion becomes very slow for increasing time  $t$ .

## 5 The adaptive Stochastic Spectral Method

In this section we introduce the aSSM. At first, a response surface in the stochastic domain of the solution of the random Van-der-Pol equation is constructed in order to compute the integrals (13). Knowing the coefficients  $\{q_i\}_{i=0}^m$  of the stochastic spectral expansion, we compute the density of  $y(t_j, p)$  at certain time steps  $t_j, j \in \mathbb{N}$ . For the selection of the time steps, we consider the magnitude of the third coefficient of the truncated stochastic spectral expansion. The density will be updated, if  $|q_2(t_j)|$  exceeds a prescribed tolerance TOL. By adapting the density we are able to reduce the approximation error, which is introduced by the truncation of the stochastic spectral expansion.

### 5.1 Computation of the response surface in random space

The response surface is constructed by computing solutions of the deterministic problem for fixed parameter values  $p_i, i = 1, \dots, n$ . Having these data points we represent the response surface by B-splines of degree 3. This approach is analogous to the stochastic collocation method, see [13, 12]. The response surfaces at time steps  $t = 0.5$  and  $t = 3.5$  are shown in figure 2.



**Figure 2:** Response surfaces of the solution process:  $y(0.5, p)$  (left) and  $y(3.5, p)$  (right) for the parameter  $p \in [1.9, 2.1]$ . The squares represent the deterministic solutions for parameters  $\{1.900, 1.933, 1.967, 2.0, 2.033, 2.067, 2.100\}$ .

The differentiation of the response surface in the representation of B-splines can be readily obtained, which will be important in section 5.2, where the adaption of the density is discussed. The response surface, once computed, serves as initial data for the computation of the response surface at the next time step. That means, that the number of deterministic evaluations at time  $t_{j+1}$  is independent of the number of deterministic evaluations at time  $t_j$ , so that expensive computations of additional solutions are not needed.

### 5.2 Computation of the density

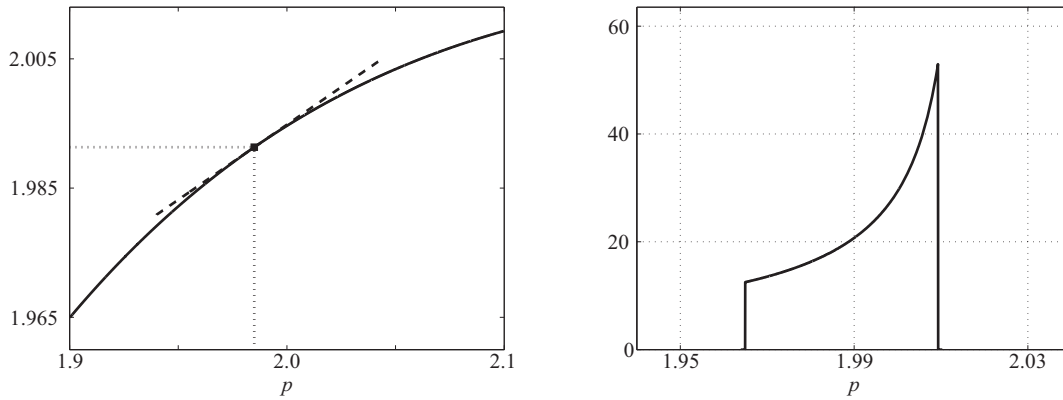
Having computed the response surface we can readily determine the density of the random variable  $y(t_j, p)$ . This method is based on the density transformation formula

$$\rho^{(j)} : \begin{cases} I & \rightarrow \mathbb{R}_0^+ \\ \hat{y} & \mapsto \sum_{r \in R} \rho^{(0)}(r) \left| \frac{d}{dp} y(t_j, p) \Big|_{p=r} \right|^{-1}, \end{cases} \quad (14)$$

with  $I \subset \mathbb{R}$  being the range of  $y$  and the sum going over all pre-images  $r \in R$  of  $y(t_j, \cdot) = \hat{y}$ , see [7]. This formula was first used in the context of stochastic spectral expansions by P. Vos in [8]. In figure 3 (left) we show the response surface at time  $t = 5.3$ . The dotted line is the tangent at point  $(1.985, 1.991)$ . By formula (14) we compute the density of the solution, which is plotted in figure 3 (right). The determination of the pre-images of  $\hat{y} \in [1.965, 2.009]$  is done by Newton's method.

The need of the determination of the pre-images of the range of  $y(t, \cdot)$  is a major drawback of the adaption of the density. In figure 4 (left) the response surface  $y(3.5, \cdot)$  is shown. At this time step the response surface has a local minimum and the pre-image computation in a neighborhood  $E$  of the local extrema, at  $p = 1.965$ , is an ill-posed problem. To handle this problem, we eliminate intervals  $\{E_i\}_{i=1}^q$  in  $I$ , where the derivative of  $y$  with respect to the random parameter  $p$ , falls below a prescribed threshold. Figure 4 (right) shows the response surface  $y(3.5, p)$  for  $p \in [1.961, 1.970]$ . The area below the dotted line will be excluded in the computation of the density. We obviously lose information by the elimination of  $\{E_i\}_{i=1}^q$ . We will address this problem in the section 6. For values in the remaining set

$$\hat{I} := \bigcup_{k=1}^n I_k := I \setminus \bigcup_{i=1}^q E_i$$

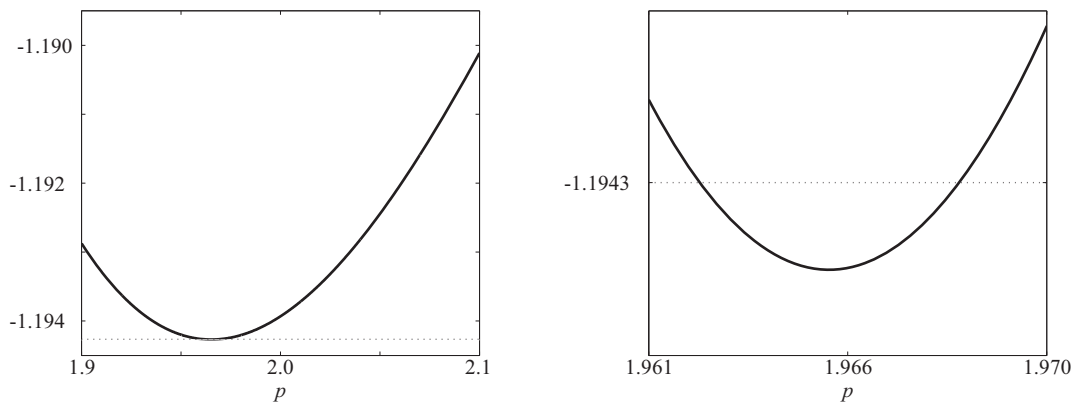


**Figure 3:** Response surface of  $y(5.3, \cdot)$  (solid line) with respective tangent at point  $(1.985, 1.991)$  (left) and weight function of the distribution of the random variable  $y(t, 5.3)$  (right).

we determine the pre-images using Newton’s method, so that we are able to compute the conditional density with respect to  $\hat{I}$ . Note, that the intervals  $\{I_k\}_{k=1}^n$ , together with the intervals  $\{E_i\}_{i=1}^q$  define a partition of  $I$ . The elimination of the intervals  $E_i$  from  $I$  is equivalent to set their masses to 0. Thereby the total mass of  $\hat{I}$ ,

$$m_T := P(\hat{I}) = \sum_{k=1}^n P(I_k),$$

is unequal to one. We correct this discrepancy by choosing the masses  $P(I_k)/m_T$  as partial masses of the sets  $I_k$ ,  $k = 1, \dots, n$ .

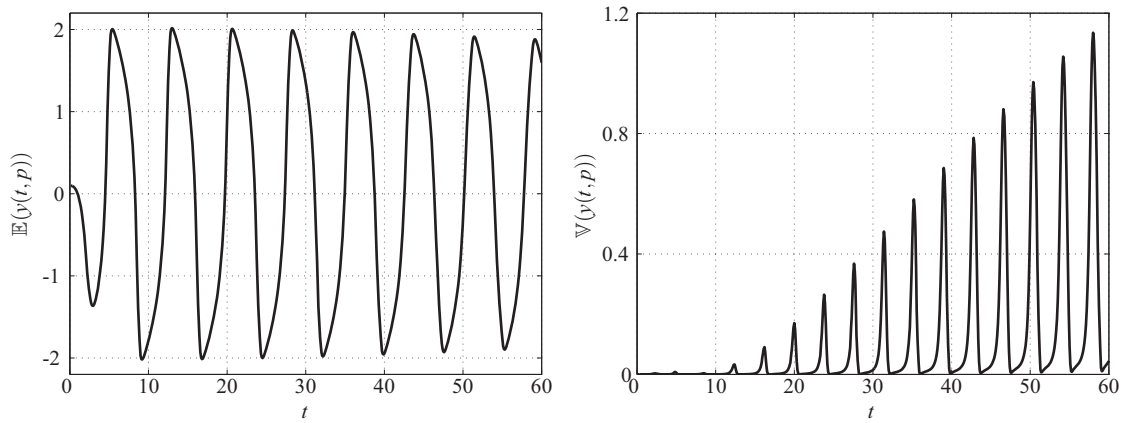


**Figure 4:** Cut-off of the area of the response surface of  $y(3.5, p)$  (solid line) where pre-image computation is an ill-posed problem (left) and magnified to the interval  $[1.961, 1.970]$  (right).

## 6 Numerical results

We return to the Wiener-Askey calculus for solving random differential equations and apply it to the random Van-der-Pol equation (12), choosing  $m = 2$  for the length of the truncated expansion. We compute the expected value and variance by the formulas from section 3. The results are shown in figure 5. In order to estimate accuracy, we compute a reference solution by Monte-Carlo simulation, using 200,000 samples. The absolute error of the variance is plotted in figure 6 (left). As we discussed in section 4, the variance approximated by Wiener-Askey calculus loses accuracy with increasing time.

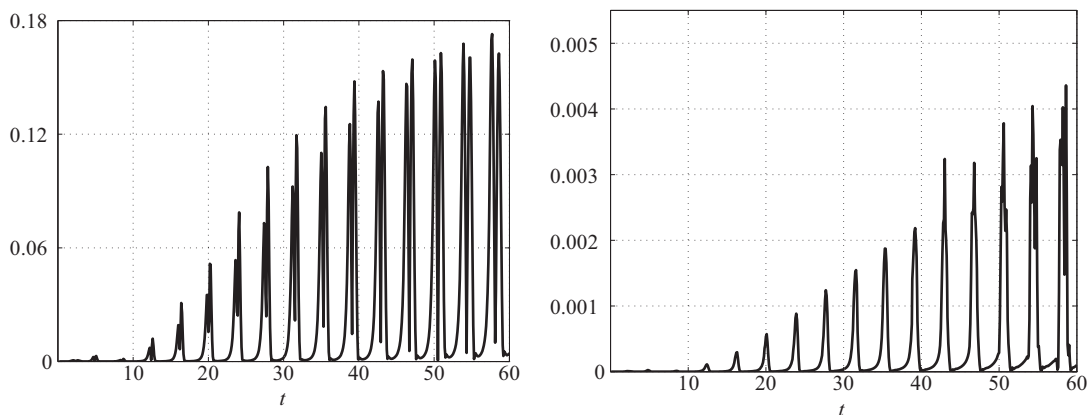
Adding more terms to the truncated series expansion does not reduce the problem. To illustrate this situation we computed the solution by Wiener-Askey calculus with 6 terms in the truncated series expansion. Although the magnitudes of the coefficients decrease with increasing polynomial order, the gradient of the descent becomes larger with increasing time. The plot in figure 7 visualizes the situation. Thus adding more terms to the truncated Wiener-Askey series is not an appropriate way to compute an accurate approximation of the solution process. The gradient of the descent of the series coefficients  $q_i$  has to be drawn into consideration as well. Instead of increasing the length of the truncated Wiener-Askey expansion, we apply aSSM to reduce the error. The choice of initializing values are step size  $h = 0.1$ ,  $\mathcal{U}[1.9, 2.1]$  for the distribution of  $p$ ,  $y(0, p) \equiv 0.1$ ,  $\dot{y}(0, p) \equiv 0.0$  and  $TOL = 10^{-5}$  (see appendix A). The associated plots of the expected value and the variance are omitted, because they differ only slightly from figure 5. The error of the variance, computed by aSSM compared to the Monte-Carlo simulation, is



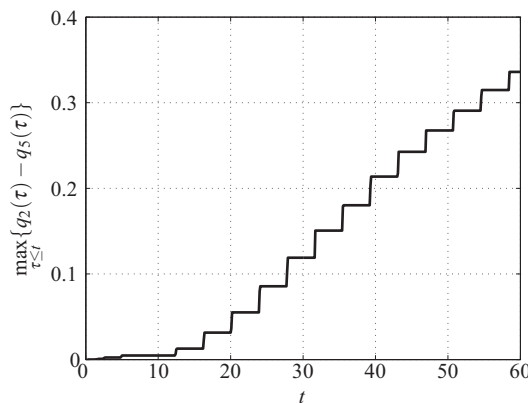
**Figure 5:** Expected value (left) and variance (right) of  $y(t, p)$  in the time interval  $[0, 60]$  computed by Wiener-Askey calculus with 3 terms in the truncated expansion.

shown in figure 6 (right). It is reduced considerably in comparison to the error of the Wiener-Askey calculus, c.f. figure 6 (left).

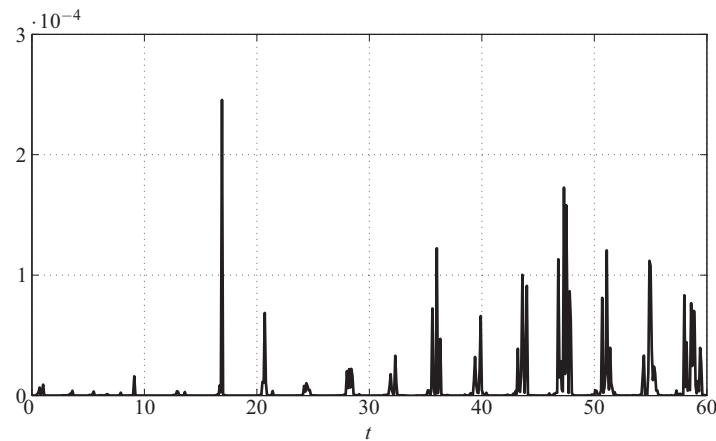
In figure 8 the arithmetic mean of the absolute values of the third coefficients of the truncated stochastic spectral expansions are plotted in the interval  $[0, 60]$ . Although these values are supposed to be below  $TOL = 10^{-5}$ , they sometimes exceed this tolerance. For example, the value  $2.5 \cdot 10^{-4}$  at time  $t = 16.9$  is much larger than  $TOL$ . This happens, because of the exclusion of the intervals  $E_i$  of the range of the solution  $y(t, \cdot)$ . If  $\hat{I} \neq I$ , the density of  $y(t, \cdot)$  is only approximated, so that the absolute value of the third coefficient of the truncated stochastic spectral expansion has to capture the offset between the approximated density and the exact density. Future work will deal with how to find a good approximation of the exact density.



**Figure 6:** Left: absolute error of variance computed by Wiener-Askey calculus in comparison to the Monte-Carlo reference solution using 200,000 samples. Right: the respective absolute error of aSSM.



**Figure 7:** Difference  $\max_{\tau \leq t} \{q_2(\tau) - q_5(\tau)\}$  between the third and last coefficient of the truncated Wiener-Askey expansion of order 5 in the time interval  $[0, 60]$ .



**Figure 8:** Arithmetic mean of the absolute values of the third coefficients of the truncated stochastic spectral expansions in the aSSM.

## 7 Conclusions and Objectives

In this article we presented the method of adaptive stochastic spectral expansion. We have shown, that aSSM is an adequate method to reduce the error arising in the long-term integration of the random Van-der-Pol equation. The standard way to capture the error, increasing the length of the truncated series expansion, is often not feasible due to the increasing computational effort and further more does not guarantee a good approximation.

aSSM is based on the adaption of the stochastic spectral expansion to the density of the solution process  $y(t, \cdot)$ . Doing so, the computational time is about one day for the problem in section 6. This is, compared to the Wiener-Askey method, which needs only an hour, a considerable drawback. But nevertheless using Monte-Carlo simulation took several weeks to approximate the solution with 200,000 samples. The specification of computational time are for a Sun Fire V40z machine with 4x Dual Core AMD Opteron(tm) Processor 875 (2,2GHz).

We see, that aSSM is a very promising method for high accuracy computation of solutions of random differential equations. The most challenging problem of this method is, that the determination of the exact density may lead to ill-posed problems. In those cases, an alternative density has to be chosen. We propose the approximation by elimination of the sets  $E_i$  as described in section 5.2. Of course, this is not the only possible choice. In order to introduce an error, which is not worse than in the Wiener-Askey calculus we can chose  $\rho^{(j)} = \rho^{(0)}$  whenever  $\hat{I} \neq I$ . To find an appropriate choice of approximation in order to minimize the truncation error is up to future work.

## 8 Acknowledgement

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## A The algorithm of aSSM

We introduce the algorithm of the adapted Stochastic Spectral Method (aSSM). It is based on a stochastic spectral expansion with adapted density:

```

INITIALIZE
  SET initial time t=0 and step size h
  SET end time T
  SET density d[0] of p
  SET initial value y[0, p]
  SET tolerance TOL
END

WHILE t < T
  t = t + h
  APPROXIMATE y[t, p] by B-splines
  COMPUTE third coefficients w.r.t. d[t - h]
  IF |q[2]| > TOL
    COMPUTE density d[t] of y[t, p]
    REDUCE support of density
    CONSTRUCT orthogonal polynomials w.r.t. d[t]
  ELSE
    d[t] = d[t - h]
  END
  COMPUTE 1st and 2nd coefficients w.r.t. d[t]
  COMPUTE expected value E[t]
  COMPUTE variance V[t]
END

```