FAULT ISOLABILITY STUDY OF NONLINEAR SYSTEMS BY REDUNDANT GRAPH

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Abstract. This paper deals with nonlinear systems and use its structure to study the diagnosis capability of simultaneous faults. The concept of redundant graph is used to study the faults diagnosis of Hessenberg systems since this concept determines subgraphs involving only known variables of the bipartite graph which can be jointed despite the faults. From this characteristic, the constraints subsystem involved in the redundant relations and the isolability conditions can be obtained.

Introduction. The interest to determine models decoupled of unknown input for nonlinear systems has been increased and one of the reason is that the fault isolation task in a diagnosis system for dynamic nonlinear systems requires observable subsystems sensitive to some faults and robust to the rest. A general theory for the nonlinear fault detection and isolation, FDI, issues is still missing and in general the solution is based on redundant information and the consistence between normal and actual process behavior. To overcome the difficulties diverse analytical tools and particular class of nonlinearities are considered, [4], [3], [1]. The conditions given with this formulation are difficult to test and satisfy in complex systems and do not help to study the additional assumptions required to get a solution.

Since to know if there is a solution, only the structure of the system plays an important role, the use of generic structural models are more appropriate for the analysis than the analytical models. Staroswiecki's school suggests the structural analysis [2] to study the fault diagnosis. This framework has two advantages: allows to cope with complex system, and requires few parameters information.

To study the faults detectability and isolability of complex process, this paper proposes the definition of *redundant graph* $\mathscr{RG}$ following the framework of Blanke's school [2]. Its advantages are shown with the issue of two faults in a upper and lower nonlinear Hessenberg System in which the absence of a solution for the diagnosis is easily justified and the candidate sensors to improve the isolability are straightforward identified.

FDI Principle. Consider the analytical model of a nonlinear process

$$\begin{aligned} \dot{x} &= f_m(x,\mathscr{F},\bar{\mathscr{F}}) + g(x,u,\mathscr{F},\bar{\mathscr{F}}) \\ y &= h(x,u,\mathscr{F},\bar{\mathscr{F}}) \end{aligned} \tag{1}$$

with $x \in \Re^n$, $u \in \Re^q$, $y \in \Re^m$, the faults set $\mathscr{F} = \{f_1, ..., f_f\}$, and the non interest faults set $\bar{\mathscr{F}} = \{\bar{f}_1, ..., \bar{f}_d\}$, the known set $\mathscr{K}$ associated to u and known variables y.

Definition 1 *Given a vector k_i integrated by a subset of known signals $\mathscr{K}_i \subset \mathscr{K}$ of system (1), the expression*

$$RR(\mathscr{K}_i) = RR(k_i, \dot{k}_i, \ddot{k}_i ...) = 0 \tag{2}$$

is called a redundancy relation RR for a set of detectable faults $\mathscr{F}$ if for all $\mathscr{K}_i$ consistent with the process free of faults, RR is zero; and if a fault $f_i \,\varepsilon\, \mathscr{F}$ occurs, RR is inconsistent or different from zero.

Thus, the faults diagnosis depend on the relations between the known variables set $\mathscr{K}$ in normal and fault conditions and a residual generator is a particular kind of a redundancy relation *RR*. The fault isolation issue is formulated as follows.

Definition 2 *Two faults sets $\mathscr{F}$ and $\bar{\mathscr{F}}$ are isolable with a known variables set $\mathscr{K}_i$ if exist at least a pair of redundant relations with different inconsistences for the two sets of faults; this means*

$$RR_k(\mathscr{K}_i)\,|_{\mathscr{F}} \neq RR_j(\mathscr{K}_i)\,|_{\bar{\mathscr{F}}}, \quad with \quad \mathscr{F} \cap \bar{\mathscr{F}} = \varnothing \tag{3}$$

If there is a inconsistent RR_i for each member of $\mathscr{F}$, the faults are concurrently isolable.

A simple way to generate the *RR*s is to link variables and constraints without numeric values by a bipartite graph and determining the relation by directed graphs with all the possible combinations of known variables $\mathscr{K}$.

Structural Analysis.

Definition 3 *Consider system (1) characterized by the set of constraints* $\mathscr{C} = \{c_1, \ldots, c_{n_c}\}$ *with* $|\mathscr{C}| = n_c = 2n + m$*, the exogenous variables* $\mathscr{U} = \{u_1, \ldots, u_q\}$*, and the endogenous* $\mathscr{Y} = \{y_1, \ldots, y_m\}$*, both known* $\mathscr{K} = \mathscr{U} \cup \mathscr{Y}$ *and the unknown variables set* $\mathscr{X} = \{x_1, \ldots, x_{2n}\}$*, including the time derivative of each state* x_i*, then (1) can be described generically by the* bipartite graph

$$\mathscr{G} = \{\mathscr{C}, \mathscr{Z}, \mathscr{E}\} \tag{4}$$

which associates the vertices $\mathscr{Z} = \mathscr{X} \cup \mathscr{K}$ *with the vertices of* $\mathscr{C}$*, by the edges* $\mathscr{E} \subset \mathscr{C} \times \mathscr{Z}$ *defined as*

$$e_{i,j} = (c_i, z_j) \quad \text{if} \quad z_j \text{ appears in } c_i \tag{5}$$

Thus, the graph $\mathscr{G}$ *can be represented by interconnection of vertices of* $\mathscr{Z}$ *and* $\mathscr{C}$ *by the edges* $\mathscr{E}$*.*

Bipartite graph are as well described by a incidence matrix where the rows are associated to the constraints c_i, the columns to the variables z_i and each edge (c_i, z_j) is indicated by 1. The evaluation of the variables $\mathscr{X}$ with the constraints $\mathscr{C}$ in a graph is equivalent to give a orientation to each edges passing by the vertices $\mathscr{X}$. Since, the key of the FDI is the redundancy between known variables and this is captured in the over-constrained part of the Dulmage-Mendelssohn decomposition $\mathscr{G}^+$, this part is only considered for FDI [2].

Definition 4 *Let* $e = (c_z, z)$ *be an edge connecting* c_z *with* z*, and consider the projection of the edge*

- *on the constraints set* $pc(e) = c_z$ *written by* $z \rightarrow c_z$ *and,*
- *on the variables set* $pz(e) = z$ *denoted by* $c_z \rightarrow z$*.*

The set of edges $\mathscr{M} \subset \mathscr{E}$ *is a matching which assigns members of the set* $\mathscr{Z}$ *with members of* $\mathscr{C}$ *if for all pair* $\{e_1, e_2\} \subset \mathscr{M}$ *such that* $e_1 \neq e_2$ *implies*

$$pc(e_1) \neq pc(e_2); \quad \text{and} \quad pz(e_1) \neq pz(e_2) \tag{6}$$

A matching graph is oriented and the arrow direction means the edge projection on the initial vertex to the final vertex. A property of the matched edges set is that any member has a common vertex.

After the matching process, the paths which connects the vertices following an orientation are equivalent to operators obtained by constraints concatenations. Then, the directed graph with initial vertices in $\mathscr{K} = \mathscr{U} \cup \mathscr{Y}$, without distinction or exogenous and endogenous variables is a *RR*. This fact allows to combine the members of $\mathscr{K}$ and to joint them looking for directed short paths to maximize the fault isolability. Thus similarly to the analytical redundancy relation, the redundant graph is defined here to search short paths between vertices of $\mathscr{K}$ in the graph context.

Definition 5 *Let* $\mathscr{K}_i := \mathscr{U}_{si} \cup y_i$ *be a subset of known variables matched with the subset of constraints* $\mathscr{C}_i$*, initial vertices of* $\mathscr{U}_{si}$ *and target vertex* y_i*, then*

$$\mathscr{RG}_i(\mathscr{C}_i; \mathscr{U}_{si}; y_i) \tag{7}$$

is a redundant graph if

- *there is consistency between the vertices of* $\mathscr{U}_{si}$ *and the target* y_i *in the paths obtained concatenating* $\mathscr{C}_i$ *without faults, and*
- *a lack of consistency between* $\mathscr{U}_{si}$ *and* y_i *in the path is produced by faults.*

Thus, the members of $\mathscr{U}_{si}$ *are considered as independent variables and are correlated with* y_i *by the* $\mathscr{RG}_i$ *and faults generated inconsistent vertices in the matching graph.*

FDI Properties of a Hessenberg Form. Consider the differential system defined on domain $\Omega \in \Re^n$, $u \in U$

$$\Sigma \begin{cases} \dot{x} = f(x,u) + F_i(x)p_1 + F_j(x)p_2 \\ y = h(x) \in \Re^2 \end{cases} \tag{8}$$

where:

1. The system is both strictly linked lower and upper Hessenberg. This means for any indexes (i, j) such that

$$j > i+1, \quad \frac{\partial f_i(x,u)}{\partial x_{j+1}} = 0, \quad \frac{\partial f_i(x,u)}{\partial x_{i+1}} \neq 0 \quad \text{and if} \quad j < i+1, \quad \frac{\partial f_i(x,u)}{\partial x_{j-1}} = 0, \quad \frac{\partial f_i(x,u)}{\partial x_{i-1}} \neq 0$$

2. The system has only 2 outputs and for any $x \in \Omega$, $y_1 = h(x_1)$ with $\frac{dh(x_1)}{dx_1} \neq 0$ and $y_2 = h(x_n)$ with $\frac{dh(x_n)}{dx_n} \neq 0$. These output properties are called upper and lower measured respectively.

$\searrow$	$x1$	x_2	x_3	x_4	x_5	x_6	x_7	$\dot{x}_1$	$\dot{x}_2$	$\dot{x}_3$	$\dot{x}_4$	$\dot{x}_5$	$\dot{x}_6$	$\dot{x}_7$	f_1	f_2	f_3
c_1	1	1						1									
c_2	1	1	1						1						1		
c_3		1	1	1						1							
c_4			1	1	1						1					1	
c_5				1	1	1						1					
c_6					1	1	1						1				1
c_7						1	1							1			
d_1	1							1									
d_2		1							1								
d_3			1							1							
d_4				1							1						
d_5					1							1					
d_6						1							1				
d_7							1							1			

Table 1: Incidence Matrix

3. Independent on the set of admissible $u(t)$, the existence of a fault p_i produces a deviation of the output such that the $\|y(t)-y_0(t)\| \neq 0$, where $y_0(t)$ is the output of the system without fault p_i.
4. The distribution vector $F_i(x)$ of the fault has $n-1$ zero elements with only the component i different from 0.

The objectives is the determination of measured variables which allow the detection and isolation of two faults. The start point of the analysis are the constraints $\mathscr{C} = \{c_1,...,c_n,d_1,...,d_n\}$, obtained from each state equation of (8) and constraints $x_{2i} := \frac{dx_i}{dt}$ called (di), the known variables $\mathscr{K} = \{y_1,y_n,u_1,u_2\}$, the unknown variables $\mathscr{X} = \{x_2,x_3,...,x_{n-1},\dot{x}_1,\dot{x}_2,...\dot{x}_n\}$, and the unknown faults $\mathscr{F} = \{p_1,p_2\}$. The incidence matrix form is shown in Table 3 for $n = 7$.

In absence of fault, the decomposition of the incidence matrix indicates that the whole system is $\mathscr{G}^+$, $|\mathscr{C}| > |\mathscr{X}|$ and diverse paths exist to link the set $\mathscr{K}$. Considering property (4) and that the faults p_1 and p_2 affect directly only the state variables pair (x_i,x_{i+k}) the five constraints associated to $(p_1,p_2$ are at most

$$
\begin{aligned}
\dot{x}_{i-1} &= f_{i-1}(x_{i-2},x_{i-1},x_i,u) && (c_{i-1})\\
\dot{x}_i &= f_i(x_{i-1},x_i,x_{i+1},u) + F_i(x)p_1 && (c_i)\\
\dot{x}_{i+1} &= f_{i+1}(x_i,x_{i+1},x_{i+2},u) && (c_{i+1})\\
\dot{x}_{i+k} &= f_{i+k}(x_{i+k-1},x_{i+k},x_{i+k3},u) + F_{i+k}(x)p_2 && (c_{i+k})\\
\dot{x}_{i+k+1} &= f_{i+k+1}(x_{i+k},x_{i+k+1},x_{i+k+2},u) && (c_{i+k+1})
\end{aligned}
$$

Moreover using property (2) of the system, the paths

$$P_1 \quad : y_1 - d_1 - \dot{x}_1 - c_1 - x_2 - d_2 - \dot{x}_2 - c_2 - x_3 ... - y_2 - d_n - \dot{x}_n \tag{9}$$

and

$$P_2 \quad : y_2 - d_n - \dot{x}_n - c_n - x_{n-1} - d_{n-1} - \dot{x}_{n-1} ... - c_2 - y_1 - d_1 - \dot{x}_1 \tag{10}$$

can be defined and if $\{u_1,y_1\}$ and y_2 and u_2 are assumed initial and target vertices respectively, the redundant graphs with $\mathscr{C}_1$ and $\mathscr{C}_2$ the constraints set associated to (9) and (10)

$$\mathscr{RG}_1(\mathscr{C}_1;u_1,y_1;y_2) \tag{11}$$

$$\mathscr{RG}_2(\mathscr{C}_2;u_1,y_1;u_2) \tag{12}$$

are generated and both are inconsistent if a fault occurs. If more than two sensors exist, the redundancy of the system is increased and the extra measures robustify the diagnosis. If fault p_1 occurs, only one constraint c_i must be eliminated from the paths and the redundancy is reduced. Assuming three initials vertices in the matching process, paths without crossing one of the even constraints can be selected. (i.e without c_i). For example if the initial set is $\mathscr{U}_{s3} = \{u_1,y_1,y_2\}$ a path without touching c_{i+2} which is affected by the fault $p2$ is obtained resulting

$$\mathscr{RG}_3(\mathscr{C} \setminus \{c_{i+2},d_{i+2}\};\mathscr{U}_{s3};u_2) \tag{13}$$

These results have been as well obtained for a pipeline model with leaks by geometric approach [6].

In the case of two Faults, since $\left|\mathscr{C}\setminus\{c_i,c_j\}\right| = |\mathscr{X}|$, there is not redundancy and the whole system cannot be evaluated; only there is a solution for particular faults configurations. To study the possibility to determine a subset, which allows to generate a residual sensitive only to one fault, one must search for a subgraphs $\mathscr{X}_m \subset \mathscr{X}$ in which a $\mathscr{RG}$ exists for each particular combination of faults.

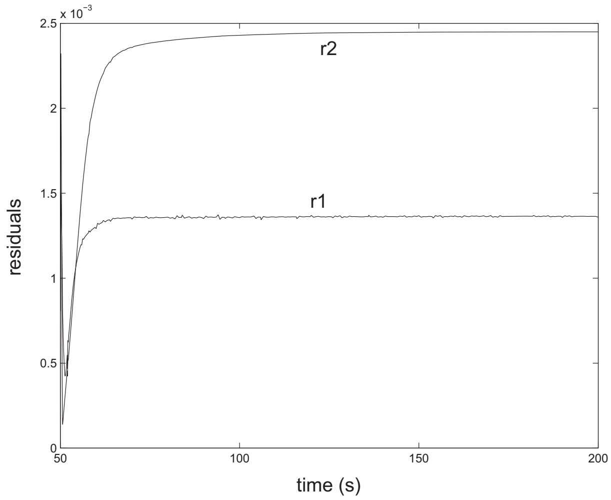

Figure 1: Residual for 2 faults

1. Faults pair getting $\mathscr{C}_f = \{c_i, c_{i+k}\}$ with $k \leq 2$. From this set of constraints and paths (9) and (10) one see that one can calculate the unknown variables until point $\dot{x}_i$ from x_1 in $P1$ and $\dot{x}_{i+1}$ from x_n in $P2$. This evaluation disregards constraints pair (c_i, c_{i+k}) and at most constraint c_{i+1}. Assuming known the value of the state x_{i+1} at the time (t_f) that the faults occur, one can evaluate

$$\frac{d\hat{x}_{i+1}}{dt} = f_{i+1}(x_i, \hat{x}_{i+1}, x_{i+2}, u) \qquad \hat{x}_{i+1}(t_f) = x_{tf}$$

increasing the cardinality of $\mathscr{X}$ and getting $\left|\mathscr{C} \backslash \mathscr{C}_f\right| = |\mathscr{X}|$.
Finally, using $\mathscr{C}_f$ one can define as a RR: $0 = -\hat{x}_i - x_i|_{P1}$, considering path $P1$ with c_i to calculate $x_i|_{P1}$ for p_1 and $0 = -\hat{x}_i - x_i|_{P2}$ considering now path $P2$ to compute $x_i|_{P2}$ for p_2.

2. Faults $\mathscr{C}_f = \{c_i, c_{i+k}\}$ with $k > 2$. There is not redundant graph and faults can be isolated. Moreover, the number of constraints associated to the faults increase and two points appears in the paths which can not be related by an intermediary constraint. Then, the faults pair cannot be isolated and new sensors, which joint the paths are required.

Then, if the states associated to the faults can be compacted in a block of the path, as case 1, one can find $\mathscr{RG}$ for the isolation issue. If the faults are not compacted in a subsystem they cannot be isolated.

Considering the fluid model in a pipeline taken from [5] divided in 3 sections with $x \in \Re^{14}$ the inconsistent RRs with the behavior of Fig. 1 for 2 faults are obtained.

Conclusions. The Redundant Graph has shown to be a useful aid to study the faults isolability of systems. In particular for a Hessenberg form, it helps to identify where sensors must be added to improve the diagnosticability to faults. Moreover one can easily determine which kind of fault are impossible to detect or to isolate from the structure of the system. This model analysis must be done before one selects the FDI procedure for a particular problem.

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