On Discrete-time Sampled-data Models for Nonlinear Systems

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Abstract. Most research in modeling and control of nonlinear systems is devoted to continuous time systems, the discrete time models being not well defined and difficult to handle. In this paper, an attempt to obtain equivalent discrete time models for sampled data nonlinear system is presented. Although the class of nonlinear systems is restricted to the so called sampled data computable nonlinear systems, it is also shown how broad the class of systems suitable to be expressed by this form is.

1 Introduction

The discrete time (DT) equivalent of a continuous time nonlinear system (CT NL) system is a problem of practical interest. In the general case, only those discretization methods based on the derivative approximation can be applied. But they are either too simple like the Euler approach or too complicated, like those based on the Runge-Kutta. In any case, they are approximations only valid for short sampling periods. Moreover, the use of approximate models for feedback control purposes should be carefully validated. In [1], the difficulties in implementing a controller being designed based on an approximate process model are illustrated. It is a common assumption that a fast sampling rate will solve the difficulties, but in this application, the SD CT controlled plant is unstable no matter how fast this sampling rate is chosen.

Recently [2], an approach for the discretization of CT NL models has been presented. The approach is only valid for models expressed as a set of ordinary differential equations affine in the input signal, and by using the so called normal form. The SD exact discretization is only achieved for a restricted class of systems but the modeling errors are bounded and related to the relative degree of the original CT system.

The purpose of this paper is to study the DT modeling of NL systems which open-loop step response is analytically computable. This class of NL systems, as shown in the paper, includes a number of other previously defined classes, like finite discretizable systems [3], and do cover a wide range of NL systems reported in the literature. A chained and modular structure is assumed in the global system. Each subsystem is nonlinear in the input, although its input may be external or coming from precedent-in-the-line subsystems. Some subsystems may have the same, different, or none external input. Our first purpose is to obtain SD NL models of these systems, that is, DT equivalent models for piecewise-constant input functions, under some special form of the nonlinearities. These DT models will allow a direct discrete simulation of the systems, as a basis for further SD NL process models.

2 Preliminaries

In this section some basic results on sampled data linear systems are summarized. The proofs are omitted due to space limitation. Assume a linear system

$$\dot{x}(t) = Ax(t) + Bu(t) \quad y(t) = Cx(t) \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the control input, $y \in \mathbb{R}^p$ is the output vector.

Lemma 2.1 *Given the linear system (1), if A is a (strictly) lower triangular matrix (A-entries are $a_{i,j} = 0, \forall j > i$), there exists a lower triangular coordinates transformation matrix T such that the transformed matrix $\bar{A}$ takes the form (2), for given β coefficients.*

$$\bar{A} = \begin{bmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ \beta_1 & \lambda_2 & 0 & \dots & 0 \\ 0 & \vdots & \vdots & \vdots & 0 \\ 0 & 0 & 0 & \beta_{n-1} & \lambda_n \end{bmatrix} \tag{2}$$

It is important to notice that the general transformed matrix $\bar{A}$ resembles a matrix in Jordan form. Nevertheless, the first subdiagonal may contain elements different from ones and zeros even if all the eigenvalues of A are equal. This form will make easier the computation of the state trajectory. The special form of (2), allows the following results:

Proposition 2.1 *For the matrix (2), the exponential* $\exp(\bar{A}t)$ *can be evaluated as*

$$e^{\bar{A}t} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ \bar{\beta}_{21}t & 1 & 0 & \dots & 0 \\ \bar{\beta}_{31}\frac{t^2}{2} & \vdots & \vdots & \vdots & 0 \\ \bar{\beta}_{n1}\frac{t^{(n-1)}}{(n-1)!} & \dots & \dots & \bar{\beta}_{n,n-1}t & 1 \end{bmatrix} e^{\,\mathrm{diag}(\bar{A})t} \text{ with } \bar{\beta}_{ij} = \prod_{k=j}^{i-1} \beta_k \;, i \geq 2 \,, j \geq 1 \,, j < i \tag{3}$$

Proposition 2.2 *For the matrix (2), the elements of* $\bar{I} \equiv \int_0^t e^{\bar{A}\tau} d\tau$ *can be evaluated as*

$$\bar{I}_{ij} = \bar{\beta}_{ij} \frac{1}{\lambda_j^{i-j+1}} + \sum_{k=0}^{i-j} (-1)^k \frac{t^{i-j-k}}{(i-j-k)!} \frac{e^{\lambda_j t}}{\lambda_j^{k+1}} \;, j < i \tag{4}$$

3 CT nonlinear in the input feedforward models

A feedforward form assumes a chained decomposition of the global system. Consider

$$\dot{x}(t) = Ax(t) + g(x(t), u(t)) \tag{5}$$

where A and $g(x,u)$ are such that, by getting the Jordan canonical form for the linear part, the state equation will be

$$\dot{z} = \bar{A}z + \bar{g}(z,u) \tag{6}$$

where $\bar{A}$, similar to (2), is a (strictly) lower triangular matrix, and the vector field $g(x,u)$ presents the triangular structure

$$\bar{g} = \begin{bmatrix} \bar{g}_1(z_1, u) \\ \bar{g}_2(z_1, z_2, u) \\ \vdots \\ \bar{g}_n(z_1, \dots, z_n, u) \end{bmatrix} \tag{7}$$

The class of systems to be considered in this paper are the so called *sampled-data computable systems* (SDC). Their structure is feedforward, with nonlinearity in the input.

A CT *nonlinear in the input feedforward system* (NIF) is defined by r blocks

$$\dot{x}_i(t) = A_i x_i(t) + B_i g_i(x_1, x_2, \dots, x_{i-1}, u_i), \quad i = 1, \cdots, r \tag{8}$$

where $x_i \in \mathbb{R}^{n_i}$ and $u_i \in \mathbb{R}^{m_i}$. The diagram of one system matching this structure is depicted in Figure 1. The ZOH

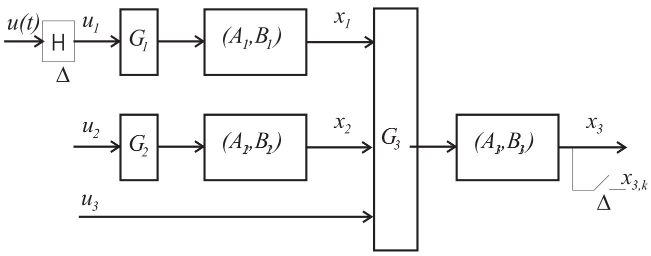

Figure 1: Nonlinear in the Input Feedforward (NIF) System

devices (as shown at input 1) will provide the inputs and the measurements will be sampled at period Δ, as shown for x_3, in order to get a sampled-data model.

The structure shown by the model (8) may be masked by an arbitrary selection of state variables. As the triangular structure should be also shown by the linear part, a similarity transformation of the state vector will enhance this property.

Proposition 3.1 *The system* Σ

$$\Sigma: \quad \dot{x}(t) = Ax(t) + g(x,u) \tag{9}$$

has a NIF structure if there exists a similarity transformation, $z(t) = Tx(t)$ *such that*

$$\Sigma: \quad \dot{z}(t) = \bar{A}z(t) + \bar{g}(z,u) \tag{10}$$

where $\bar{A} = TAT^{-1}$ *is an r-block triangular matrix, i.e.,* $A_{i,j} = 0; \quad i > j, \quad i = 1, 2, \dots, r$ *and the entries of the vector field* $\bar{g}(z,u)$ *also exhibit the triangular structure, i.e.,* $\bar{g}_i(z,u) = \bar{g}_i(z_1, z_2, \dots, z_{i-1}, u)$, *where* $z_i(t)$ *is the subset of the state vector attached to the i-block. The similarity transformation, if it exists, is not unique.*

The triangular structure may be hidden in the nonlinear term. If this is the case, it will be possible to find a diffeomorphism $z = h(x)$, such that the new state equation is NIF.

4 Main result

Let us consider a nonlinear in the input system (8). For those subsystems with only external inputs, as the ZOH devices keep constant their value in the intersampling time, the SD model will be linear in the state. Therefore, similarly to the linear case

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bg(u_k)d\tau; \tag{11}$$

Thus, the SD model will be

$$x_{k+1} = A_d x_k + B_d g(u_k) \tag{12}$$

where $g(u_k)$ is the constant value of the input contribution to the state derivative, in this time interval. It is worth to remark that the elements of both matrices are generalized exponential functions, as it appears at (3), and (4). Therefore, the **exact** SD model of the nonlinear in the input system (8) can be computed as for linear systems, keeping the same structure and nonlinearity in the input.

For those subsystems with inputs coming from other subsystems (see Figure 1), denoted as *inner* subsystems, the input is not constant anymore in between sampling times. In order to be able to compute the exact SD model for a NIF system, the following assumption is required.

Assumption 1. Given a NIF system (8), for any inner subsystem, i, the nonlinear function $g_i(x_1(t), x_2(t), \ldots, x_{i-1}(t), u_i(t))$ is a polynomial function in the state variable arguments.

Under assumption 1, $u_i(t) = u_i(k\Delta)$, $\forall t \in [\Delta k, \Delta(k+1))$, being constant in the sampling period, the nonlinear function $g_i(t, u_i(k\Delta))$ will be also a vector field of generalized exponential functions, easy to compute. This allows to state the following result.

Theorem 4.1 *The exact SD model for a generic NL system represented in the NIF structure, such as (8)*

$$\begin{aligned} \dot{x}_i(t) &= A_i x_i(t) + B_i g_i(x_1, x_2, \ldots, x_{i-1}, u_i), \\ & i = 1, \cdots, r \end{aligned}$$

*where $g_i(x_1(t), x_2(t), \ldots, x_{i-1}(t), u_i(t))$ is a **polynomial** function in the state variable arguments, can be easily computed, their coefficients being obtained after matrix exponential computations similar to (11), (3), and (4).*

Example. Consider *The plate and ball system*, [4]. The locally modeled kinematics of a ball rolling in a plate are:

$$\begin{aligned} \dot{x}_1 &= u_1; \quad \dot{x}_2 = u_2; \quad \dot{x}_3 = x_1 u_2 - x_2 u_1; \\ \dot{x}_4 &= x_3 u_1; \quad \dot{x}_5 = x_3 u_2 \end{aligned} \tag{13}$$

It can be exactly integrated to get the SD equivalent model:

$$\begin{aligned} x_{1,k+1} &= x_{1k} + \Delta u_{1k}; \qquad x_{2,k+1} = x_{2k} + \Delta u_{2k}; \qquad x_{3,k+1} = x_{3k} + \Delta(x_{1k}u_{2k} - x_{2k}u_{1k}) \\ x_{4,k+1} &= x_{4k} + \Delta x_{3k}u_{1k} + \frac{\Delta^2}{2}(x_{1k}u_{2k} - x_{2k}u_{1k})u_{1k}; \qquad x_{5,k+1} = x_{5k} + \Delta x_{3k}u_{2k} + \frac{\Delta^2}{2}(x_{1k}u_{2k} - x_{2k}u_{1k})u_{2k} \end{aligned}$$

In order to enlarge the class of systems, two assumptions are removed. First, the nonlinearities do not need to be polynomial, but keeping the properties of being continuously differentiable. Second, the triangular structure condition is not required.

Consider the CT NI system (8) such as $\dot{x}(t) = Ax(t) + g(x(t), u(t))$, the SD equivalent model should be computed from

$$x_{k+1} = e^{A\Delta}x_k + \int_0^{\Delta} e^{A\tau} g(x(t-\tau), u_k)\, d\tau \tag{14}$$

But the integral should be approximated, as $x(t-\tau)$ is not constant in the integration interval. The most simple approach consists on considering the approximated sampled-data model

$$x_{k+1} = e^{A\Delta}x_k + \int_0^{\Delta} e^{A\tau} d\tau\, g(x_k, u_k) \tag{15}$$

The approximation error between the true state and the computed one can be estimated as follows. Assume g is locally Lipschitz, then for $\tau \in [0, \Delta)$ one has $\|g(\hat{z}_k) - g(z(k\Delta + \tau))\| \leq L(\|\tilde{z}_k\| + \|z_k - z(k\Delta + \tau)\|)$ for some constant L (in particular, $L = 0$ if the system is linear, with $g = B$ constant). Continuity of the solution $z(\tau)$ on the compact $[k\Delta, (k+1)\Delta]$ ensures $\|z_k - z(k\Delta + \tau)\| \leq L_z\tau$. Thus, considering that the state is measured at each sampling period $\tilde{z}_k = 0$, the prediction error for the state at $(k+1)\Delta$ is bounded by

$$\left\|\tilde{z}_{(k+1)\Delta}\right\| \leq LL_z u_k \int_0^{\Delta} \left\|e^{\bar{A}\tau}\right\| \tau d\tau \tag{16}$$

This, in the general case, will give an error of $O(\Delta^3)$, as easily seen from the results in propositions 2.1 and 2.2. Alternatively, with similar assumptions about the function $g(x,u)$, it can be expanded in polynomial form and apply the Theorem 4.1 to a truncated series of polynomial terms. The SD model will be also an approximated model.

5 Comparative example

In order to evaluate the proposed approach, the SD model of a NL system is computed and compared with the model obtained by using the Euler discretization as well as the approach proposed in [2]. This imposes the constraint of selecting an input affine system convertible into the feedback normal form. Thus, consider the system

$$\Sigma : \begin{cases} \dot{x}_1 = -x_1 + u_1 \\ \dot{x}_2 = -2x_2 - u_2 \\ \dot{x}_3 = x_3 + x_1 + x_2 u_2 \end{cases} \tag{17}$$

showing the NIF structure. The exact SD model is

$$x_{1,k+1} = a_{11}x_{1,k} + b_{11}u_{1,k}; \quad x_{2,k+1} = a_{22}x_{2,k} + b_{22}u_{2,k};$$

$$x_{3,k+1} = a_{31}x_{1,k} + a_{33}x_{3,k} + b_{31}u_{1,k} + b_3 x_{2,k}u_{2,k} + b_{32}u_{2,k}^2$$

with $a_{11} = e^{-\Delta}, b_{11} = 1 - e^{-\Delta}, a_{22} = e^{-2\Delta}, b_{22} = 0.5(1 - e^{-2\Delta})$, $a_{31} = \Delta, a_{33} = e^{\Delta}, b_3 = 1 - e^{-\Delta}, b_{31} = -1 - \Delta + e^{\Delta}, b_{32} = -1/2(2 - e^{\Delta} - e^{-\Delta})$. In Figure 2, from $x_0 = [2 \quad 3 \quad 4]^T$, the x_3 state variable response for constant inputs, ($u_1(t) = 1, \forall t \geq 0$, $u_2(t) = -4, \forall t \geq 0.4s$) computed by the exact solution and the different approximated ones, with sampling period $\Delta = 0.1$ s, are shown.

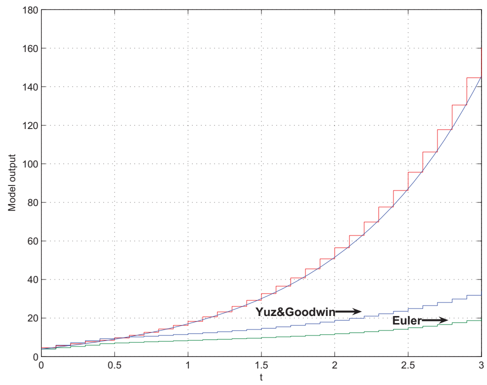

Figure 2: State variable x_3 time response

6 Conclusions

In this paper a procedure to obtain a DT model for a SD NL system is presented. The DT equivalent model is exact (giving the same input step response) for a given class of NL system, those denoted as nonlinear in the input feedforward systems. This equivalence is independent of the sampling period. Many practical NL systems can be expressed in this form, as illustrated in several examples. For any softly NL system, an approximate model is proposed, the approximation error being dependent on the sampling period.

7 References

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