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STATE ESTIMATION IN CONTROL PROBLEMS UNDER UNCERTAINTY AND NONLINEARITY

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Abstract. The paper is devoted to state estimation problems for control systems described by nonlinear differential equations with quadratic nonlinearity. The topics of the paper come from the theory of dynamical systems with unknown but bounded uncertainties related to the case of set-membership description of uncertainty. Applying results of the theory of trajectory tubes of differential control systems with uncertain parameters and techniques of ellipsoidal estimation theory developed for linear control systems we present new approaches that allow to find external and internal set-valued estimates for trajectory tubes in new classes of uncertain control system. Numerical simulation results illustrating the theoretical approaches are also included.

1 Introduction

The topics of this paper come from the control theory for systems with unknown but bounded uncertainties related to the case of set-membership description of uncertainty [2, 7, 8, 9, 10, 11]. The motivations for these studies come from applied areas ranged from engineering problems in physics to economics as well as to ecological modelling. The paper presents recent results in the theory of tubes of solutions (trajectory tubes) to differential control systems modelled by nonlinear differential inclusions with uncertain parameters or functions.

We will start by introducing the following basic notations. Let R^n be the n–dimensional Euclidean space and $x'y$ be the usual inner product of $x, y \in R^n$ with the prime as a transpose, with $\| x \| = (x'x)^{1/2}$. Denote comp R^n to be the variety of all compact subsets $A \subseteq R^n$ and conv R^n to be the variety of all compact convex subsets $A \subseteq R^n$.

Consider the control system described by the ordinary differential equation

$$\dot{x} = f(t, x, u(t)), \;\; t \in [t_0, T] \tag{1}$$

with function $f : T \times R^n \times R^n \to R^m$ measurable in t and continuous in other variables. Here x stands for the state vector, t stands for time and control $u(\cdot)$ is a measurable function satisfying the constraints

$$u(\cdot) \in U = \{u(\cdot) : u(t) \in U_0, \;\; t \in [t_0, T]\}, \;\; U_0 \in \text{comp} R^m. \tag{2}$$

Let us assume that the initial condition $x(t_0)$ to the system (1) is unknown but bounded

$$x(t_0) = x_0, \;\; x_0 \in X_0 \in \text{comp} R^n. \tag{3}$$

Let absolutely continuous function $x(t) = x(t, u(\cdot), t_0, x_0)$ be a solution to (1) with initial state x_0 satisfying (3) and with control function $u(t)$ satisfying (2). The differential system (1)–(3) is studied here in the framework of the theory of uncertain dynamical systems (differential inclusions [5, 1]) through the techniques of trajectory tubes [10]

$$X(\cdot) = \bigcup \{ \, x(\cdot) = x(\cdot, u(\cdot), t_0, x_0) \mid x_0 \in X_0, \; u(\cdot) \in U \, \} \tag{4}$$

which combine all solutions $x(\cdot, u(\cdot), t_0, x_0)$ to (1)- (3).

One of the main problems of the theory of uncertain systems consists in describing and estimating the trajectory tube $X(\cdot)$ of the nonlinear system (1)- (3). The point of special interest is to find the t – cross-section $X(t)$ of $X(\cdot)$ which is actually the attainability domain (reachable set) of the control system (1)–(3) at the instant t. The set $X(t)$ may be considered also as the set-valued estimate of the unknown state $x(t)$ of the uncertain dynamical system if we will treat the control functions in (1)- (3) as unknown but bounded disturbances.

Let us mention here the well-known result [5] from the theory of differential inclusion that the trajectory tube $X(\cdot)$ coincides with the set of all solutions $\{x(\cdot) = x(\cdot, t_0, x_0)\}$ to the following differential inclusions

$$\dot{x} \in F(t, x) = \bigcup \{ \, f(t, x, u) \;\mid\; u \in U_0 \, \}, \; t \in [t_0, T] \tag{5}$$

with the initial condition similar to (3)

$$x(t_0) = x_0, \;\; x_0 \in X_0. \tag{6}$$

So we will use further the same notation $X(\cdot)$ for both trajectory tubes either for the control system (1)–(3) or for the differential inclusion (5)–(6).

It should be noted that the exact description of reachable sets $X(t)$ of a control system is a difficult problem even in the case of linear dynamics. The estimation theory and related algorithms basing on ideas of construction outer and inner set-valued estimates of reachable sets have been developed in [11, 2] for linear control systems.

In this paper the modified state estimation approaches which use the special quadratic structure of nonlinearity of studied control system and use also the advantages of ellipsoidal calculus [11, 2] are presented. We develop here new ellipsoidal techniques related to constructing external and internal set-valued estimates of reachable sets and trajectory tubes of the nonlinear system. Some estimation algorithms basing on combination of discrete-time versions of evolution funnel equations and ellipsoidal calculus [11, 2] are given. Examples and numerical results related to procedures of set-valued approximations of trajectory tubes and reachable sets are also presented. The applications of the problems studied in this paper are in guaranteed state estimation for nonlinear systems with unknown but bounded errors and in nonlinear control theory.

2 Problem statement

Let us describe the modified state estimation approaches for the special class of nonlinear uncertain control systems. Consider the case when the right-hand side $f(t,x,u(t))$ in (1) is quadratic in state variable x and is linear in control variable u. So we will study the problems of control and state estimation for a dynamical control system of the following type

$$\dot{x}(t) = A(t)x(t) + g(x(t)) + G(t)u(t), \;\; t_0 \le t \le T, \tag{7}$$

with unknown but bounded initial condition

$$x(t_0) = x_0, \;\; x_0 \in X_0, \tag{8}$$

$$u(t) \in U_0, \;\; t \in [t_0, T]. \tag{9}$$

Here matrices $A(t)$ and $G(t)$ (of dimensions $n \times n$ and $n \times m$, respectively) are assumed to be continuous on $t \in [t_0,T]$, X_0 and U_0 are compact and convex. The nonlinear n-vector function $g(x)$ in (7) is assumed to be of quadratic type

$$g(x) = (g_1(x), \dots, g_n(x)), \;\; g_i(x) = x'B_i x, \; i = 1, \dots, n, \tag{10}$$

where B_i is a constant $n \times n$ - matrix ($i = 1, \dots, n$).

Consider the following differential inclusion [5] related to (7)–(9) (with $P(t) = G(t)U_0$)

$$\dot{x}(t) \in A(t)x(t) + f(x(t)) + P(t), \;\; t \in [t_0,T], \;\; x(t_0) = x_0 \in X_0. \tag{11}$$

We introduce here the following additional notations. Denote as $B(a,r)$ the ball in R^n, $B(a,r) = \{x \in R^n : \| x - a \| \le r\}$, I is the identity $n \times n$-matrix. Denote by $E(a,Q)$ the ellipsoid in R^n, $E(a,Q) = \{x \in R^n : \; (Q^{-1}(x-a),(x-a)) \le 1\}$ with center $a \in R^n$ and symmetric positive definite $n \times n$–matrix Q. For any $n \times n$-matrix Q denote its track as $\mathrm{Tr}\, Q$ and its determinant as $|Q|$. Denote as $h(A,B)$ the Hausdorff distance for A, $B \subseteq R^n$, $h(A,B) = \max\{h^+(A,B),\, h^-(A,B)\}$, with $h^+(A,B)$ and $h^-(A,B)$ being the Hausdorff semidistances between A and B, $h^+(A,B) = \sup\{d(x,B) \mid x \in A\}$, $h^-(A,B) = h^+(B,A)$, $d(x,A) = \inf\{\|x-y\| \;\mid y \in A\}$.

Assume now that $P(t) = E(a,Q)$ in (11), matrices B_i ($i = 1,...,n$) are symmetric and positive definite, $A(t) \equiv A$. We may assume that all trajectories of the system (11) belong to a bounded domain $D = \{x \in R^n : \| x \| \le K\}$ where the existence of such constant $K > 0$ follows from classical theorems of the theory of differential equations and differential inclusions [5].

The main problem is to construct external and internal set-valued estimates of reachable sets $X(t)$ of the nonlinear system (7)–(9). The approach presented here uses the techniques of ellipsoidal calculus developed for linear control systems. It should be noted that external ellipsoidal approximations of trajectory tubes may be chosen in various ways and several minimization criteria are well-known. We consider here the ellipsoidal techniques related to construction of external estimates with minimal volume (details of this approach and motivations for linear control systems may be found in [2, 11]).

3 External estimates of reachable sets

3.1 First approach

From the structure (10) of the function g we have two auxiliary results. Their proofs are based on algebraic properties of quadratic forms and are omitted here.

Lemma 1 *The following estimate is true*

$$\| g(x) \| \leq N, \qquad N = K^2 (\sum_{i=1}^{n} \lambda_i^2)^{1/2}, \tag{12}$$

where λ_i is the maximal eigenvalue for matrix B_i ($i = 1, ..., n$).

Lemma 2 *For all $t \in [t_0, T]$ the inclusion*

$$X(t) \subset X^*(t)$$

holds where $X^*(\cdot)$ is a trajectory tube of the linear differential inclusion

$$\dot{x} \in Ax + B(c, \sqrt{n}N/2), \;\; x_0 \in X_0, \tag{13}$$

where $c = \{N/2, \ldots, N/2\} \in R^n$ and number N is defined in (12).

The following theorem gives the external estimate of the trajectory tube $X(t)$ of the differential inclusion (11).

Theorem 1 *Let $X_0 = B(0, r)$, $r \leq K$ and $t_* = min\, \{\frac{K-r}{\sqrt{2M}}\; ; \; \frac{1}{L}\; ; \; T\}$. Then for all $t \in [t_0, t_*]$ the following inclusion is true*

$$X(t, t_0, X_0) \subset E(a^+(t), Q^+(t)), \tag{14}$$

where

$$M = K\sqrt{\lambda} + N + P, \;\; P = \sqrt{\sum_{i=1}^{n} a_i^2} + \sqrt{\tilde{\lambda}}, \;\; L = \sqrt{\lambda} + 2K\sqrt{\sum_{i=1}^{n} \lambda_i^2},$$

with λ, λ_i and $\tilde{\lambda}$ being the maximal eigenvalues of matrices AA', B_i $(i = 1, ..., n)$ and Q respectively, and vector function $a^+(t)$ and matrix function $Q^+(t)$ satisfy the equations

$$\dot{a}^+ = Aa^+ + a + c, \;\; a^+(t_0) = 0 \tag{15}$$

$$\begin{aligned} &\dot{Q}^+ = AQ^+ + Q^+A^T + qQ^+ + q^{-1}Q^*, \;\; q = \{n^{-1}Tr((Q^+)^{-1}Q^*)\}^{1/2}, \\ &Q^+(t_0) = Q_0 = r^2 I. \end{aligned} \tag{16}$$

Here

$$Q^* = (p^{-1} + 1)\tilde{Q} + (p+1)Q, \;\; \tilde{Q} = \frac{nN^2}{2} I, \tag{17}$$

and p is the unique positive solution of the equation

$$\sum_{i=1}^{n} \frac{1}{p + \alpha_i} = \frac{n}{p(p+1)}, \tag{18}$$

with $\alpha_i \geq 0$ $(i = 1, ..., n)$ being the roots of the following characteristic equation

$$| \tilde{Q} - \alpha Q | = 0. \tag{19}$$

Proof. Applying Lemmas 1-2 and the ellipsoidal techniques [2, 11], and comparing the inclusions (11) and (13) we come to the relation (14).

Example 1. Consider the following control system

$$\begin{cases} \dot{x}_1 &= 6x_1 + u_1, \\ \dot{x}_2 &= x_1^2 + x_2^2 + u_2, \end{cases} \quad 0 \leq t \leq T, \tag{20}$$

$$X_0 = B(0, 1), \; P(t) = B(0, 1), \; T = 0.15, \; K = 2.6. \tag{21}$$

Results of computer simulations based on the above theorem for this system are given at Fig. 1.

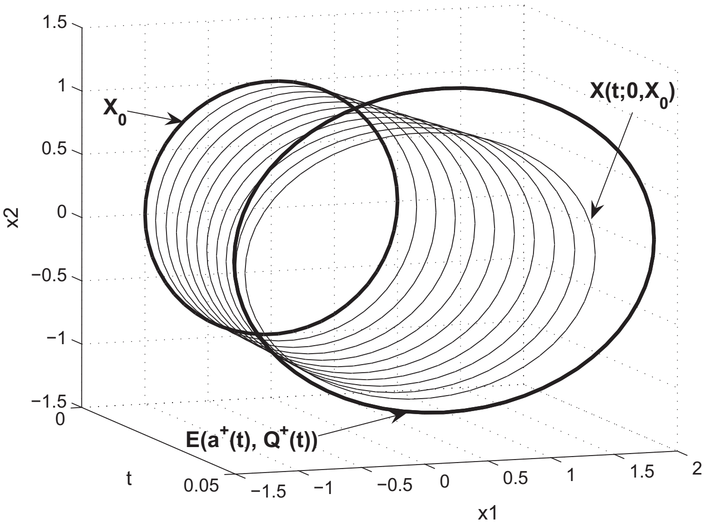

Figure 1: Reachable sets $X(t,t_0,X_0)$ and their estimates $E(a^+(t),Q^+(t))$ (here $t_* = \sqrt{2}/29.2$).

3.2 Second approach

The approach discussed here is related to evolution equations of the funnel type that describe the dynamics of set–valued system states $X(t)$ of the differential inclusion (5)–(6). The basic assumptions on set–valued map $F(t,x)$ for the following results to be true may be found in [10, 12, 15].

Let us consider the "equation" for $X(t)$ (the funnel equation),

$$\lim_{\sigma\to+0} \sigma^{-1} h(\, X(t+\sigma), \bigcup_{x\in X(t)} (x+\sigma\, F(t,x))\,) = 0, \;\; t\in[t_0,T], \tag{22}$$

with the initial set-valued condition

$$X(t_0) = X_0. \tag{23}$$

Theorem 2 ([12, 15]) *The trajectory tube $X(t)$ of the system (5)–(6) is the unique set-valued solution to the evolution equation (22)-(23).*

Other versions of funnel equations may be considered by substituting the Hausdorff distance h for a semidistance h^+ [11]. The solution to the h^+-versions of the evolution equation may be not unique and the "maximal" one (with respect to inclusion) is studied in this case. Mention here also the second order analogies of funnel equations for differential inclusions and control systems based on ideas of Runge-Kutta scheme [4, 13, 14]. Discrete approximations for differential inclusions based on set-valued Euler's method were developed in [4, 3].

Let us discuss the estimation approach based on techniques of evolution funnel equations. Consider the following uncertain system

$$\dot{x} \in Ax + \tilde{f}(x)d + E(\hat{a},\hat{Q}),\; x_0\in X_0 = E(a_0,Q_0),\; t_0\le t\le T, \tag{24}$$

where $x\in R^n$, $\|x\|\le K$, d is a given n-vector and a scalar function $\tilde{f}(x)$ has a form $\tilde{f}(x) = x'Bx$ with a symmetric and positive definite matrix B.

Note that the direct application of funnel equations for finding trajectory tubes $X(t)$ is very difficult because it takes a huge amount of computations based on grid techniques. The following theorem related to our special case of nonlinearity presents an easy computational tool to find estimates of $X(t)$ by step-by-step procedures. For a simpler case of system nonlinearities the approach was presented in [6].

Theorem 3 *Let $X_0 = E(a,k^2B^{-1})$ with $k\neq 0$. Then for the trajectory tube $X(t)$ of the system (24) and for all $\sigma>0$ the following inclusion holds*

$$X(t_0+\sigma) \subseteq E(a^+(\sigma),Q^+(\sigma)) + o(\sigma)B(0,1), \;\; \lim_{\sigma\to+0}\sigma^{-1}o(\sigma) = 0, \tag{25}$$

where

$$a^+(\sigma) = a(\sigma)+\sigma\hat{a},\;\; Q^+(\sigma) = (p^{-1}+1)Q(\sigma) + (p+1)\sigma^2\hat{Q}, \tag{26}$$

$$a(\sigma) = a + \sigma(Aa + a'Ba \cdot d + k^2 d), \;\; Q(\sigma) = k^2(I + \sigma R)B^{-1}(I + \sigma R)', \; R = A + 2dd'B, \tag{27}$$

number p is the unique positive solution of the equation

$$\sum_{i=1}^{n} \frac{1}{p + \lambda_i} = \frac{n}{p(p+1)}, \tag{28}$$

and $\lambda_i \geq 0$ are the roots of the equation $|Q(\sigma) - \lambda\sigma^2\hat{Q}| = 0$.

Proof. The proof of this result is similar to the proof of Theorem 3 [6].

Based on this result we may formulate the following scheme that gives the external estimate of trajectory tube $X(t)$ of the system (24) with given accuracy.

Algorithm of external estimating. Subdivide the time segment $[t_0, T]$ into subsegments $[t_i, t_{i+1}]$ where $t_i = t_0 + ih$ $(i = 1, \ldots, m)$, $h = (T - t_0)/m$, $t_m = T$. Repeat consequently the following steps, at the end of the process we will get the external estimate: $E(a(t), Q(t))$ of the tube $X(t)$ with accuracy tending to zero when $m \to \infty$.

- Given $X_0 = E(a, k_0^2 B^{-1})$ with $k_0 \neq 0$, define $X_1 = E(a_1, Q_1)$ from Theorem 3 for $a_1 = a^+(\sigma)$, $Q_1 = Q^+(\sigma)$, $\sigma = h$.
- Define k_1^2 as the maximal eigenvalue of the matrix $B^{1/2}Q_1B^{1/2}$.
- Consider the system on the next subsegment $[t_1, t_2]$ with $E(a_1, k_1^2 B^{-1})$ as the initial ellipsoid at instant t_1.

Note that k_1 defined at second step of the algorithm is the smallest positive constant such that $E(a_1, Q_1) \subset E(a_1, k_1^2 B^{-1})$.

Example 2. Consider the following system

$$\begin{cases} \dot{x_1} &= -x_1, \\ \dot{x_2} &= \frac{1}{2}x_2 + 3(\frac{x_1^2}{4} + x_2^2), \end{cases} \qquad X_0 = E(0, Q_0), \;\; Q_0 = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}. \tag{29}$$

Here $d = \{0, 3\}$, $\tilde{f}(x) = x'Bx$ with $B = Q_0^{-1}$, $T = 0.15$. Results of computer simulations based on Theorem 3 are shown at Fig. 2.

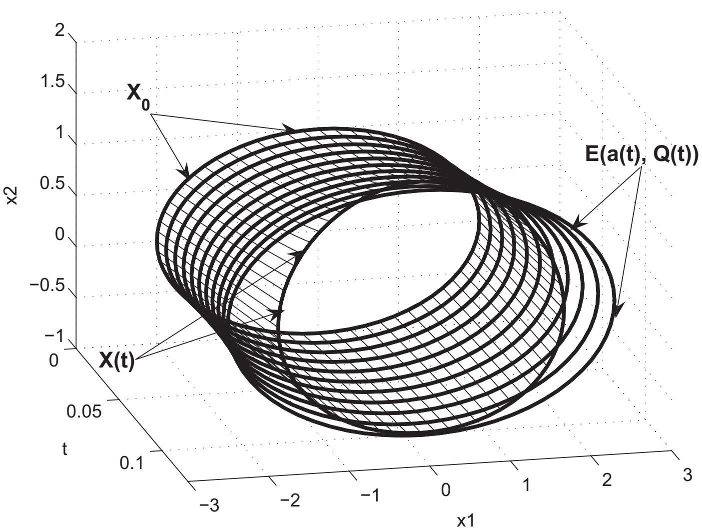

Figure 2: Trajectory tube $X(t)$ and its external ellipsoidal tube $E(a(t), Q(t))$.

4 Internal estimates of reachable sets

Consider now the internal set-valued estimates of reachable sets $X(t)$ of the uncertain nonlinear system (24). As in the previous section we formulate first the following auxiliary result.

Theorem 4 *Let $X_0 = E(a, k^2B^{-1})$ with $k \neq 0$. Then for the trajectory tube $X(t)$ of the system (24) and for all $\sigma > 0$ the following inclusion holds*

$$E(a^-(\sigma), Q^-(\sigma)) \subseteq X(t_0 + \sigma) + o(\sigma)B(0,1), \;\; \lim_{\sigma \to +0} \sigma^{-1}o(\sigma) = 0, \tag{30}$$

where

$$a^-(\sigma) = a(\sigma) + \sigma\hat{a},$$
$$Q^-(\sigma) = Q(\sigma) + \sigma^2\hat{Q} + 2\sigma Q(\sigma)^{1/2}(Q(\sigma)^{-1/2}\hat{Q}Q(\sigma)^{-1/2})^{1/2}Q(\sigma)^{1/2}, \quad (31)$$

and $a(\sigma)$, $Q(\sigma)$ *are defined in (27).*

Proof. The proof of this result is similar to the proof of Theorem 3 [6].

Based on this result we may formulate the following scheme that gives the internal estimate of trajectory tube $X(t)$ of the system (24).

Algorithm of internal estimating. Subdivide the time segment $[t_0, T]$ into subsegments $[t_i, t_{i+1}]$ where $t_i = t_0 + ih$ $(i = 1, \dots, m)$, $h = (T - t_0)/m$, $t_m = T$. Repeat consequently the following steps, at the end of the process we will get the external estimate: $E(a(t), Q(t))$ of the tube $X(t)$ with accuracy tending to zero when $m \to \infty$.

- Given $X_0 = E(a, k_0^2 B^{-1})$ with $k_0 \neq 0$, define $X_1 = E(a_1, Q_1)$ from Theorem 4 for $a_1 = a^-(\sigma)$, $Q_1 = Q^-(\sigma)$, $\sigma = h$.
- Define k_1^2 as the minimal eigenvalue of the matrix $B^{1/2}Q_1B^{1/2}$.
- Consider the system on the next subsegment $[t_1, t_2]$ with $E(a_1, k_1^2 B^{-1})$ as the initial ellipsoid at instant t_1.

Note that k_1 defined at second step of the algorithm is the largest positive constant such that $E(a_1, k_1^2 B^{-1}) \subset E(a_1, Q_1)$.

Example 3. Consider the following system

$$\begin{cases} \dot{x}_1 &= x_1 + x_1^2 + x_2^2 + u_1, \\ \dot{x}_2 &= -x_2 + u_2, \end{cases}, \quad 0 \le t \le T. \quad (32)$$

Here $t_0 = 0$, $T = 0.3$, $h = 0.025$, $a_0 = \hat{a} = (0,0)$, $Q_0 = \hat{Q} = I$. Results of computer simulations based on Theorem 4 are shown at Fig. 3.

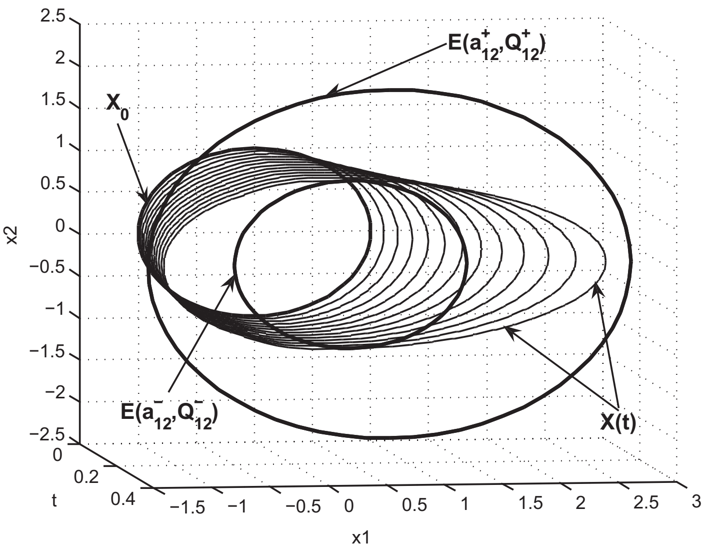

Figure 3: Trajectory tube $X(t)$ and its external and internal ellipsoidal estimates $E(a^+(t), Q^+(t))$, $E(a^-(t), Q^-(t))$.

5 Conclusion

The paper deals with the problems of control and state estimation for a dynamical control system described by differential inclusions with unknown but bounded initial state. The solution to the differential system is studied through the techniques of trajectory tubes with their cross-sections $X(t)$ being the reachable sets at instant t to control system.

Basing on the well-known results of ellipsoidal calculus developed for linear uncertain systems we present the modified state estimation approaches which use the special nonlinear structure of the control system and simplify calculations. Examples and numerical results related to procedures of set-valued approximations of trajectory tubes and reachable sets were also presented.

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