I. Troch, F. Breitenecker, eds. ISBN 978-3-901608-35-3

Time-optimal Swing-up and Damping Feedback Controls of a Nonlinear Pendulum

F. Chernousko, S. Reshmin
Institute for Problems in Mechanics of the Russian Academy of Sciences, Russia

Corresponding author: S. Reshmin,
Institute for Problems in Mechanics of the Russian Academy of Sciences
Vernadsky Ave. 101-1, Moscow 119526, Russia, reshmin@ipmnet.ru

Abstract. Two time-optimal control problems for the nonlinear pendulum are solved. The pendulum is a classical nonlinear system that often serves as a test model in nonlinear dynamics and control theory. We assume that the bounded control torque is applied to the axis of the pendulum. The terminal state is either the upper unstable or the lower stable equilibrium position of the pendulum; thus, we study the time-optimal swing-up and damping control problems, respectively. The peculiarity of these problems is that the pendulum has a cylindrical phase space and an infinite number of equivalent equilibrium positions which differ by 2π. The feedback controls for both the swing-up and the damping cases have a very complicated structure, which is obtained numerically for a wide range of the system parameters.

1 Introduction

In this paper, we obtain a time-optimal control in the feedback form that steers a pendulum to the upper unstable or to the lower stable equilibrium position. The absolute value of the control torque is constrained. The solution is based on the maximum principle [6].

It is known that the presence of nonlinearity in the equation of motion of the pendulum results in a periodic structure (cylindrical property) in the angle of the synthesis pattern. An infinite set of terminal points in the state space corresponds to the upper or lower equilibrium position.

The cylindrical property of the state space results in specific features of the feedback control. The main specific feature is the presence of a separation, or dispersal, curve on the cylinder such that two optimal trajectories with the same motion time start from each point of this curve.

For a large control torque, we can omit the nonlinear term in the equation of motion of the pendulum. The feedback control pattern in this case consists of parabolic switching curves passing through the terminal points and separation curves arranged between them. The equation of the separation curves in the case of a large control torque can be obtained in an explicit form [3].

The question arises: how the feedback control pattern changes when the maximum admissible control torque gradually decreases beginning with a certain sufficiently large value? The answer is given below. The results are based on the earlier papers by the authors [2, 7, 8, 9], where a time-optimal feedback control was designed for steering a nonlinear pendulum to the upper unstable or lower stable equilibrium position. The solution was given for various values of the maximum possible control torque.

2 Statement of the problem

Consider a pendulum that is able to rotate about a horizontal axis O and is controlled by a torque M applied to it. We use the following notation: φ is the angle between the pendulum and the vertical axis, m is the pendulum mass, J is its moment of inertia relative to the axis O, l is the distance from the axis O to the center of mass of the pendulum, and g is the gravitational acceleration.

The equation of motion of the pendulum has the form

$$J\ddot{\varphi} + mgl\sin\varphi = M, \tag{1}$$

where the dot denotes the derivative with respect to time t. The control torque is subjected to the constraint

$$|M| \le M_0, \tag{2}$$

where M_0 is a given constant. Introduce dimensionless variables

$$t' = \omega t \quad \left(\omega = \sqrt{\frac{mgl}{J}}\right), \qquad x_1 = \varphi, \qquad x_2 = \frac{d\varphi}{dt'}, \qquad u = \frac{M}{M_0}, \qquad k = \frac{M_0}{mgl} \tag{3}$$

to represent equation (1) in the following form:

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -\sin x_1 + ku. \tag{4}$$

The dot denotes the derivative with respect to the dimensionless time t'. Below, we will omit $'$ after t'. Constraint (2) becomes

$$|u(t)| \leq 1. \tag{5}$$

The initial conditions for system (4) are arbitrary

$$x_1(0) = x_1^0, \qquad x_2(0) = x_2^0, \tag{6}$$

and the terminal coordinates correspond to the upper unstable or lower stable equilibrium positions:

$$x_1(T) = \pi + 2\pi n, \qquad x_2(T) = 0, \qquad T \to \min \qquad \text{(swing-up control)}, \tag{7}$$

$$x_1(T) = 2\pi n, \qquad x_2(T) = 0, \qquad T \to \min \qquad \text{(damping control)}, \tag{8}$$

where n is an arbitrary integer.

The controls that satisfy constraint (5) for all $t \in [0, T]$ and steer system (4) from an arbitrary state (6) to a terminal state (7) (in the case of swing-up control problem) or (8) (in the case of damping control problem) in a minimum possible time are found in a feedback form.

3 The maximum principle

Following the maximum principle [6], we introduce the Hamiltonian for system (4)

$$H = p_1 x_2 + p_2(-\sin x_1 + ku). \tag{9}$$

Here, p_1 and p_2 are the adjoint variables that satisfy the equations

$$\dot{p}_1 = p_2 \cos x_1, \qquad \dot{p}_2 = -p_1. \tag{10}$$

The optimal control satisfying constraint (5) is determined by the condition

$$u = \operatorname{sign} p_2. \tag{11}$$

It follows from equations (7) (or (8)), (9), and (11) that at the terminal time instant $t = T$ we have

$$H_T = k|p_2(T)| \geq 0.$$

This inequality is one of the necessary optimality conditions [6].

It follows from equation (11) that the optimal control takes on the values $u = \pm 1$. To obtain it in the feedback form, it is sufficient to find the switching curves and dispersal curves in the plane $x_1 x_2$ confining the domains where $u = +1$ and $u = -1$. There are no singular controls in our optimal control problem, see Chapter 7 of [5].

Note that the switching curves consist of the points where the control $u(t)$ changes in sign during the motion along the optimal trajectory. The dispersal curves are generated by the points at which the optimal control can be equal to either $+1$ or -1, and two optimal trajectories starting at each of these points reach the terminal state (7) or (8) in the same time.

In terms of the field of optimal trajectories, the difference between the switching curves and dispersal curves is as follows: the optimal trajectories may start from the dispersal curves, but cannot arrive at them. In addition, the optimal trajectories cannot coincide with the dispersal curves. At the same time, the phase point of the system can arrive at a switching curve, move along it, or start from it.

Below, we present the field of optimal trajectories constructed numerically for a wide range of the parameter k. The dispersal and switching curves are designated by thick and less thick lines, respectively. The optimal trajectories are depicted by thin lines. The arrows indicate the direction of time growth along the trajectories.

All feedback control patterns present a part of the phase plane bounded either by two *main switching curves* or by two *main dispersal curves*. Each of these curves is symmetric about the point of its intersection with axis x_1. The distance between two points of intersection is equal to 2π. The complete phase portrait can be obtained by a translation of the presented segment to the left or right by the quantity $2\pi n$, $n = \pm 1, \pm 2, \ldots$.

4 Swing-up control

In Figures 1 and 2, all bounded switching curves touch the abscissa axis at one of their ends, and each of the dispersal curves divides the optimal trajectories from different families (corresponding to different n).

The figures show that, when k increases, the number of switching curves and dispersal curves decreases gradually. After passing the threshold value $k \approx 0.80$, we have the simplest structure of optimal feedback control. In this case,

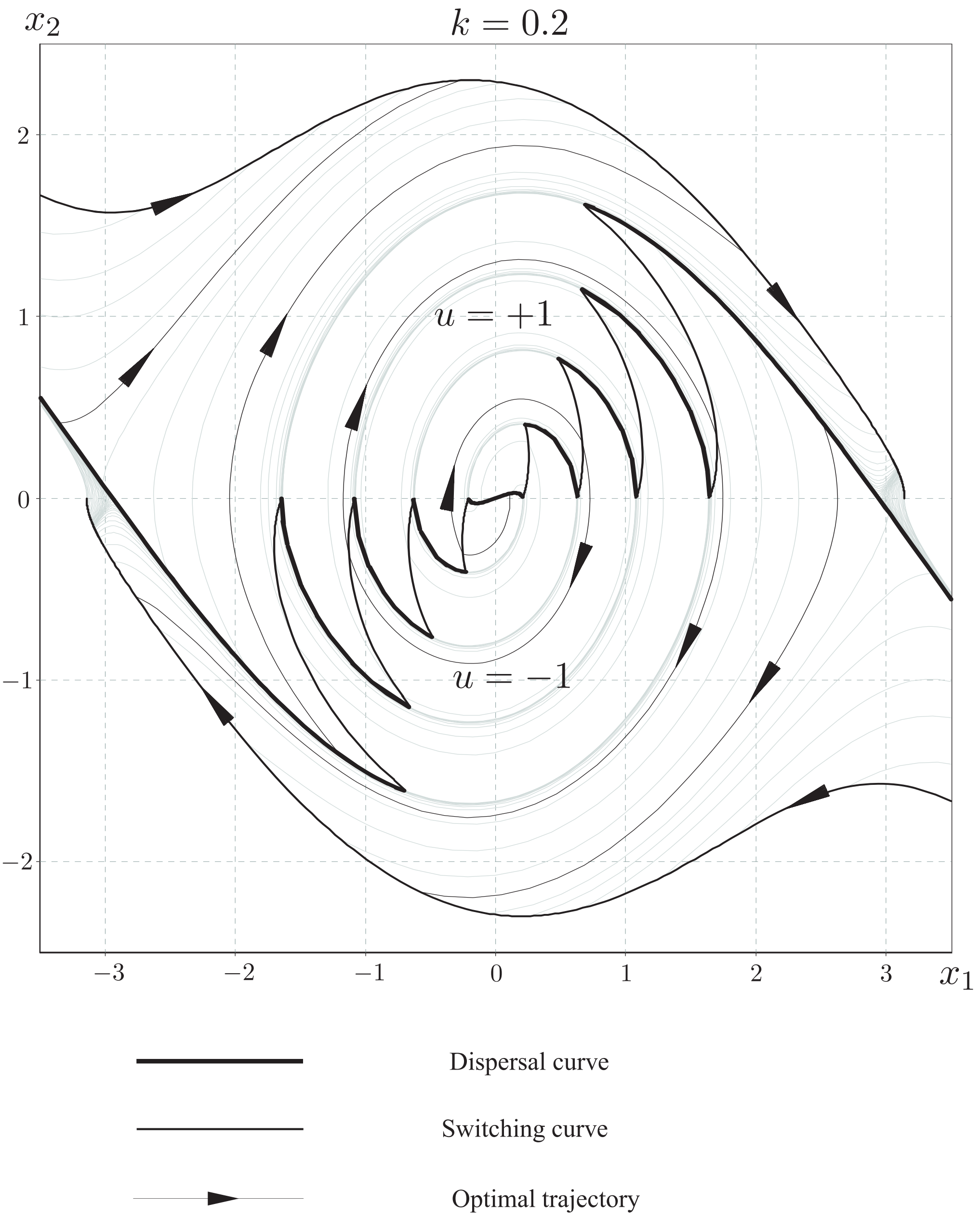

Figure 1: The time-optimal swing-up feedback control for $k = 0.2$.

the pattern contains the main switching curves and the smooth dispersal curve passing between them. The specified mechanism of transformation of the phase portrait corresponding to the optimal feedback control is depicted on a larger scale in Figure 3 (the main switching curves are not shown).

Let us describe the properties of these phase portraits. Note that, in Figure 3, the switching curves do not touch the abscissa axis, and the dispersal curve passing through the point $(0,0)$ has three clearly distinguishable smooth legs. This is explained by the fact that its middle leg separates the optimal trajectories that belong to two different families, and any of the extreme legs separates optimal trajectories of the same family. A similar pattern takes place in a domain close to the origin and in other ranges of k when the number of switching curves and dispersal curves changes (for example, see Figure 4, where the threshold value $k \approx 0.44$).

Specific behavior of switching curves and dispersal curves for small k is represented in Figure 5.

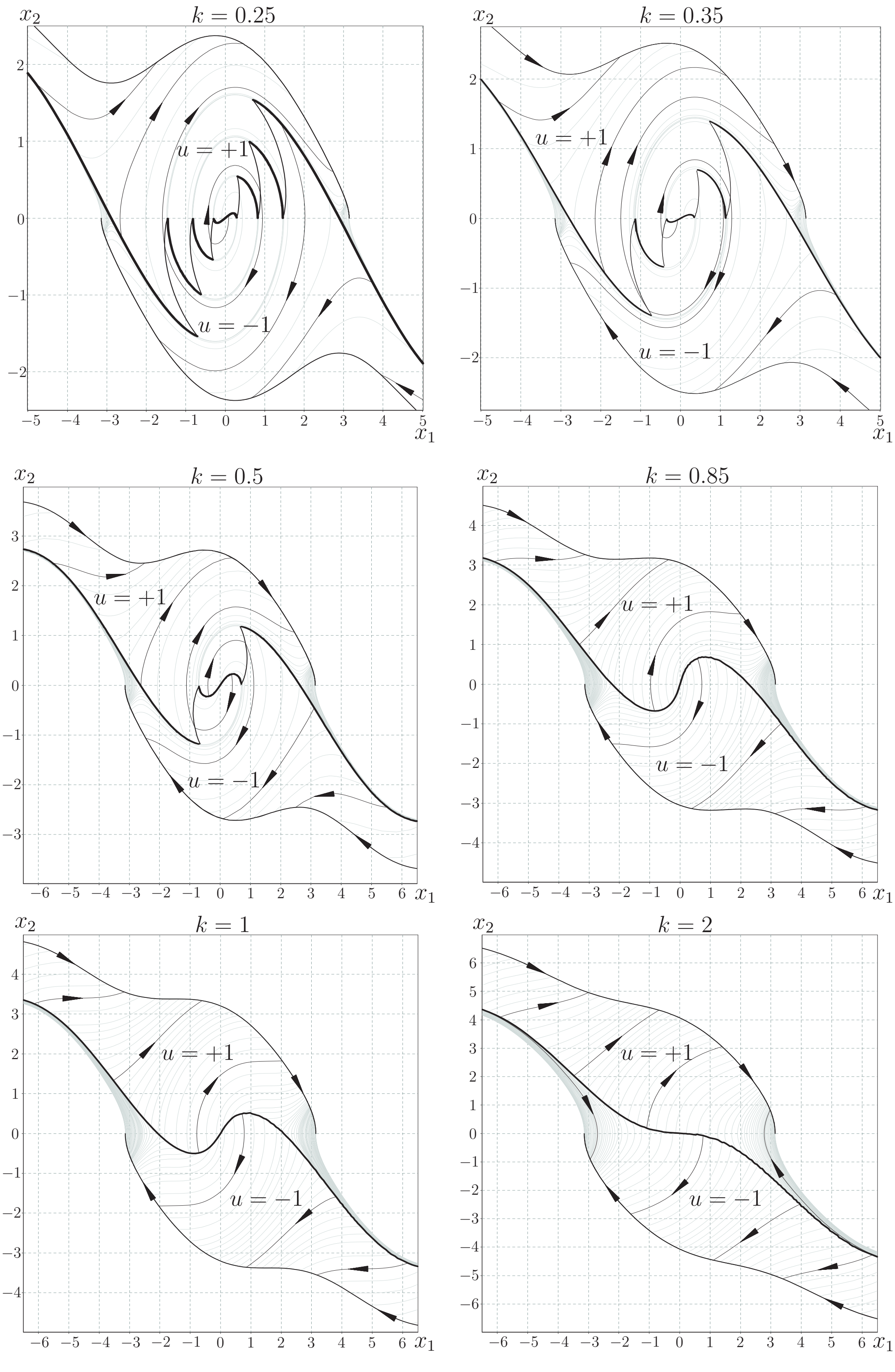

Figure 2: The time-optimal swing-up feedback control for $k = 0.25$, 0.35, 0.5, 0.85, 1, and 2.

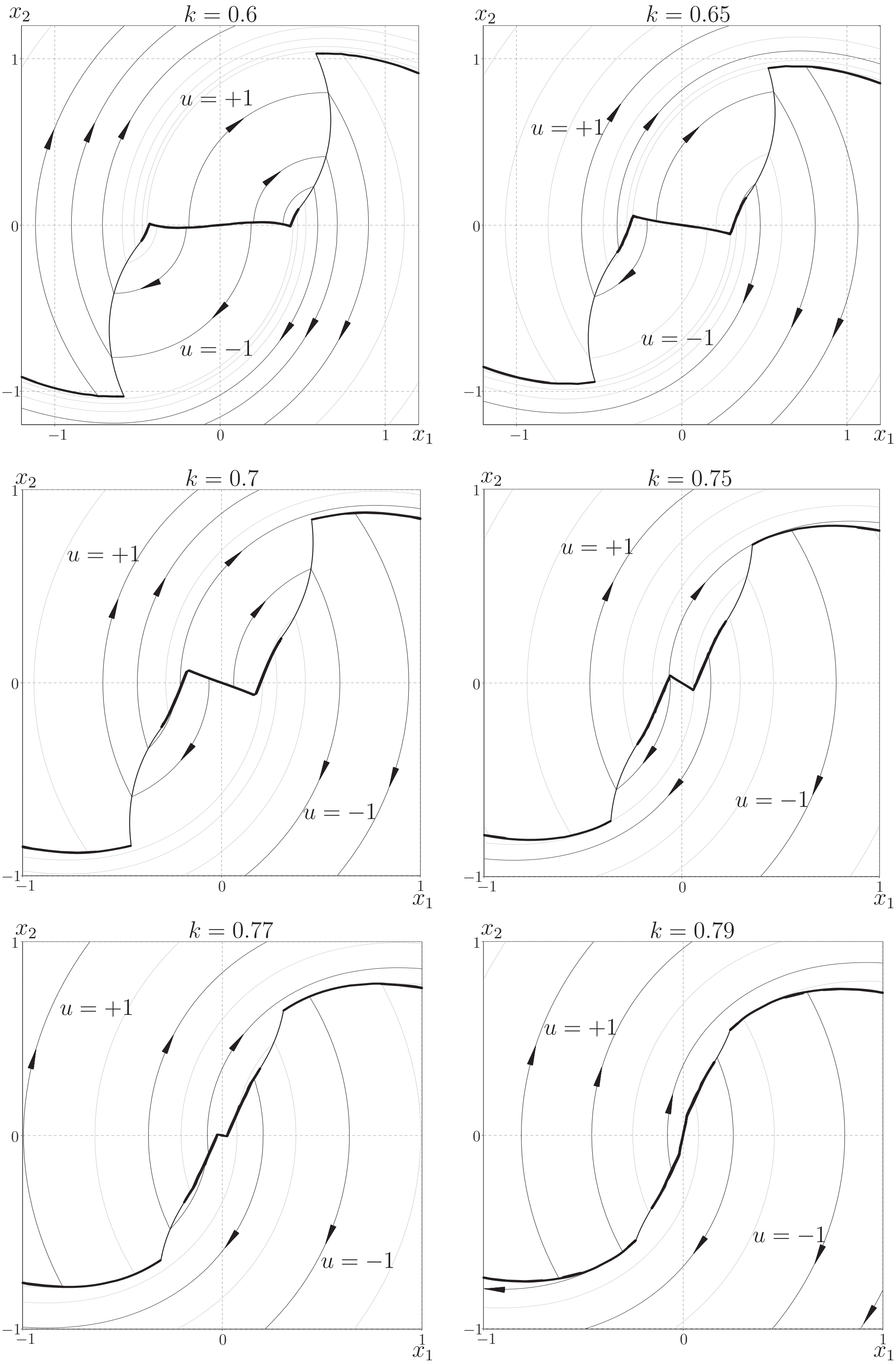

Figure 3: The transition to a unique (smooth) dispersal curve when *k* increases, $k = 0.6$, 0.65, 0.7, 0.75, 0.77, and 0.79.

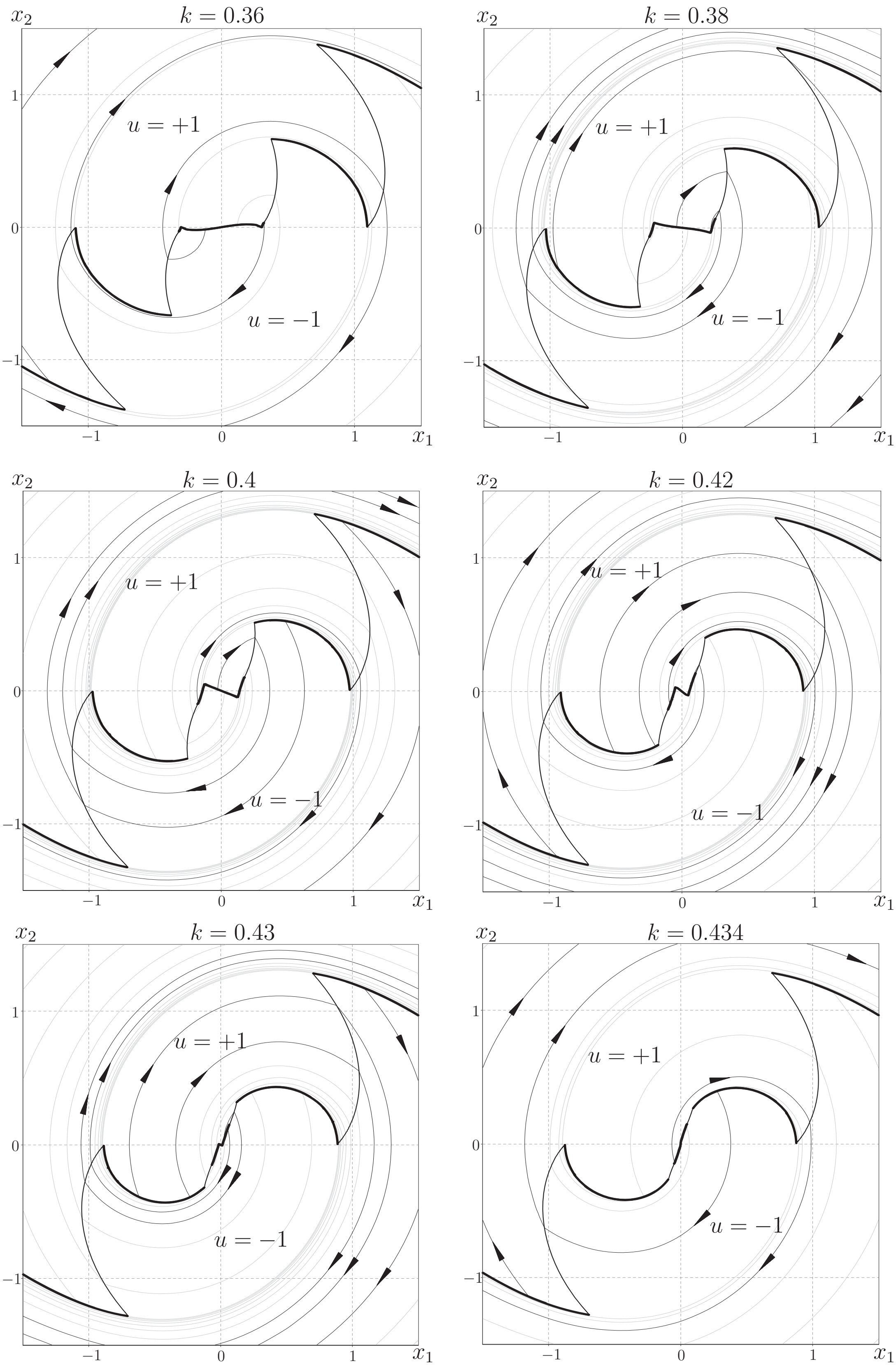

Figure 4: The transformation of the phase portrait when *k* increases, $k = 0.36$, 0.38, 0.4, 0.42, 0.43, and 0.434.

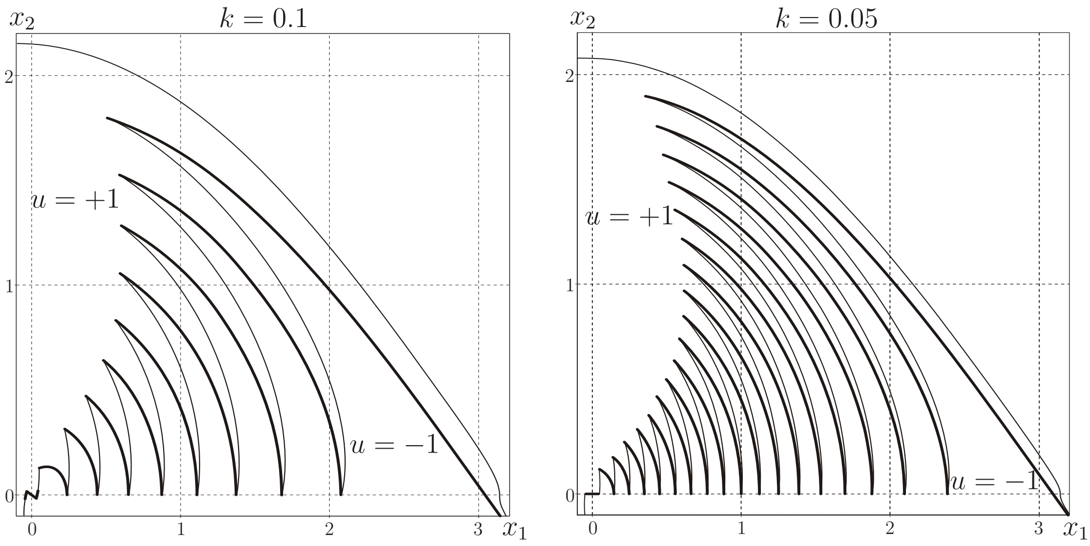

Figure 5: The optimal feedback control in the first quadrant of the phase plane for $k = 0.1$ and 0.05.

5 Damping control

Figures 6–9 show the formation of the breaks of the main switching curve passing through the terminal point $(0,0)$. The calculations have shown that the first of these breaks as *k* decreases appears as a result of transformation of the boundaries of the so-called FLAG domains, which are bounded by the arcs of switching curves and dispersal curves. The existence of the FLAG domains is closely related to the generation of the optimal trajectories with two switching instants of the control. In Figures 6–9, some areas of the feedback control pattern are denoted by a rectangle, as well as shown separately with a larger scale.

Remark. Here, we use the term FLAG introduced in [1], which is formed by the initial letters of the names of the authors of [4]. Paper [4] analyzes control problem for an equation that is more general than equations (4). However, in that paper, a time-optimal steering from any point of the phase plane to the origin was considered, and the cylindrical property of the phase space was not taken into account. The presence of an infinite number of FLAG domains in the phase plane for large controls is the most essential feature found in [4].

In Figure 6 ($k = 1.036, 1.01$), we can see the boundary part of the FLAG domain, which does not touch the main switching curve. The comparison of feedback control patterns for $k = 1.036$ and $k = 1.01$ allows one to make the conclusion that the location of the right boundary of the FLAG domain is sensitive to the parameter *k*.

Figures 7 ($k = 1$, 0.85, 0.8) and 8 ($k = 0.75$, 0.65, 0.64, 0.62) show the process of generation of the first break on the main switching curve. For $k = 0.85$, the FLAG domain and the main switching curve merge generating a sufficiently long “slot", which is much shorter for $k = 0.8$, and for $k = 0.75$, 0.65, and 0.64, it transforms into a sharp “tooth", which is turned so that its peak touches the abscissa axes for $k = 0.62$. While the dispersal curve, which generates the initial bottom boundary of the FLAG domain, disappears completely.

Further generation of breaks (as the control torque decreases) occurs in a similar way. Figure 9 ($k = 0.31$, 0.305, 0.3) illustrates the formation of the third break of the main switching curve.

6 Conclusions

In this paper, the feedback time-optimal control for steering a nonlinear pendulum to the upper unstable and lower stable equilibrium position is presented. Fine details of the structure of the field of optimal trajectories are found and investigated, including the behavior of switching curves and dispersal curves. The feedback control patterns obtained by numerical methods show the process of transformation of the structure of these curves as the magnitude of the control torque changes. In particular, we can track how the breaks of the switching curves (in the case of the damping control) arise in the transition from the case of large control torques, where these curves are smooth, to the case of small control torques corresponding to switching curves with breaks.

7 Acknowledgements

This work was supported by the Russian Science Support Foundation, by the Program for Supporting Leading Scientific Schools in Russia (project NSH-4315.2008.1), and the Russian Foundation for Basic Research (projects 07-01-92109, 08-01-00411, 08-08-00292).

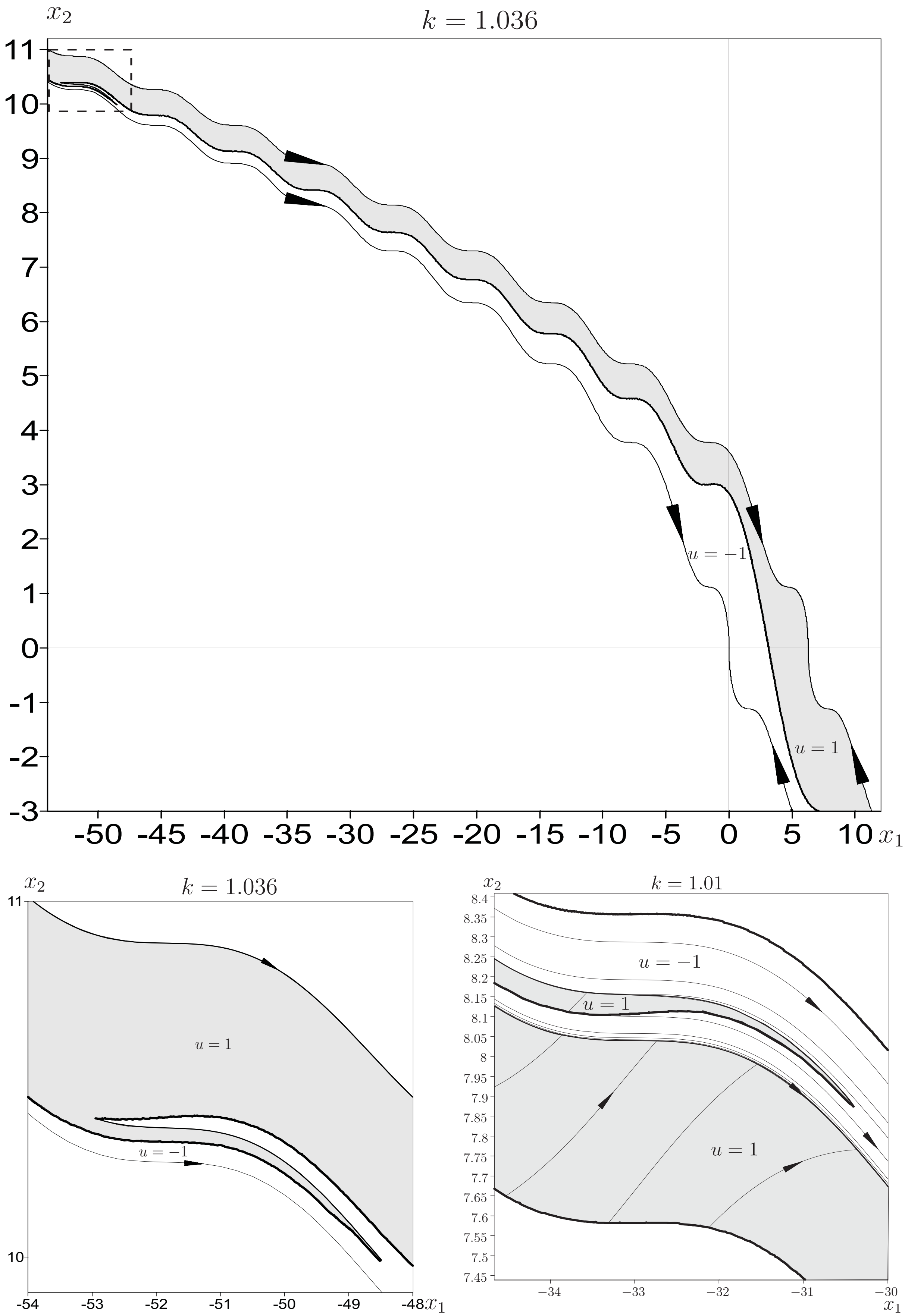

Figure 6: The time-optimal damping feedback control for $k = 1.036$ and 1.01.

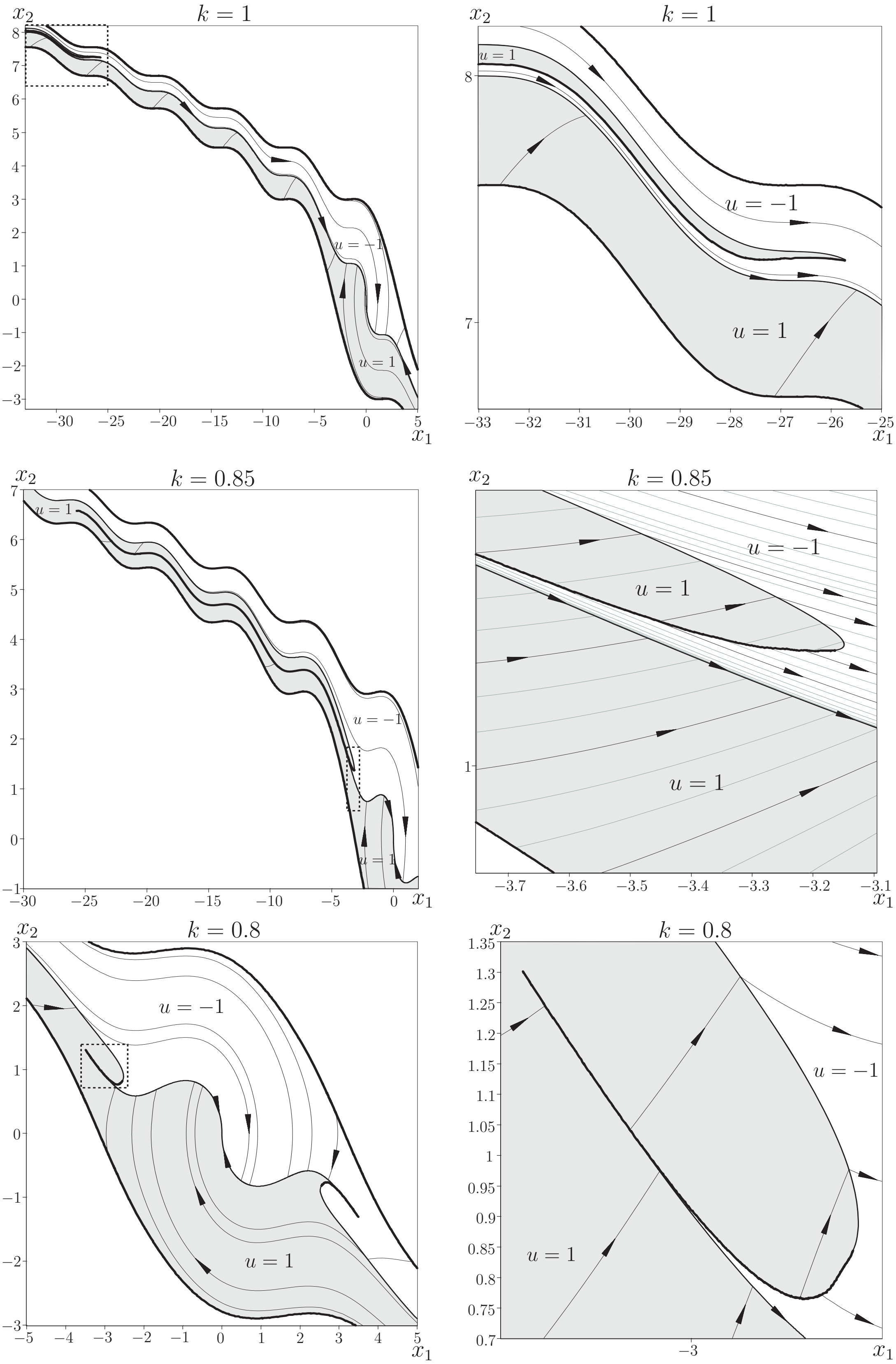

Figure 7: The time-optimal damping feedback control for $k = 1$, 0.85, and 0.8.

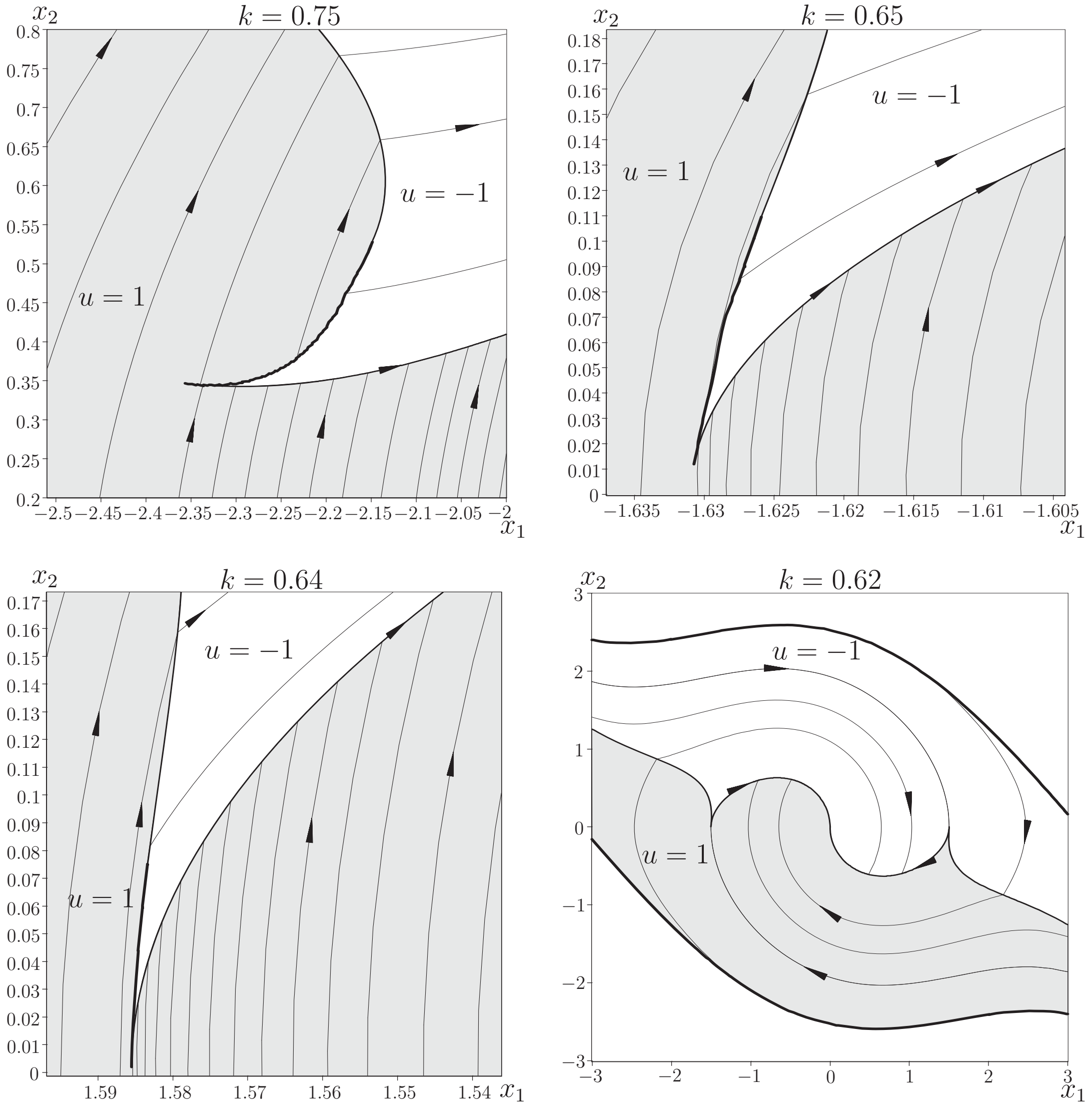

Figure 8: The time-optimal damping feedback control for $k = 0.75$, 0.65, 0.64, and 0.62.

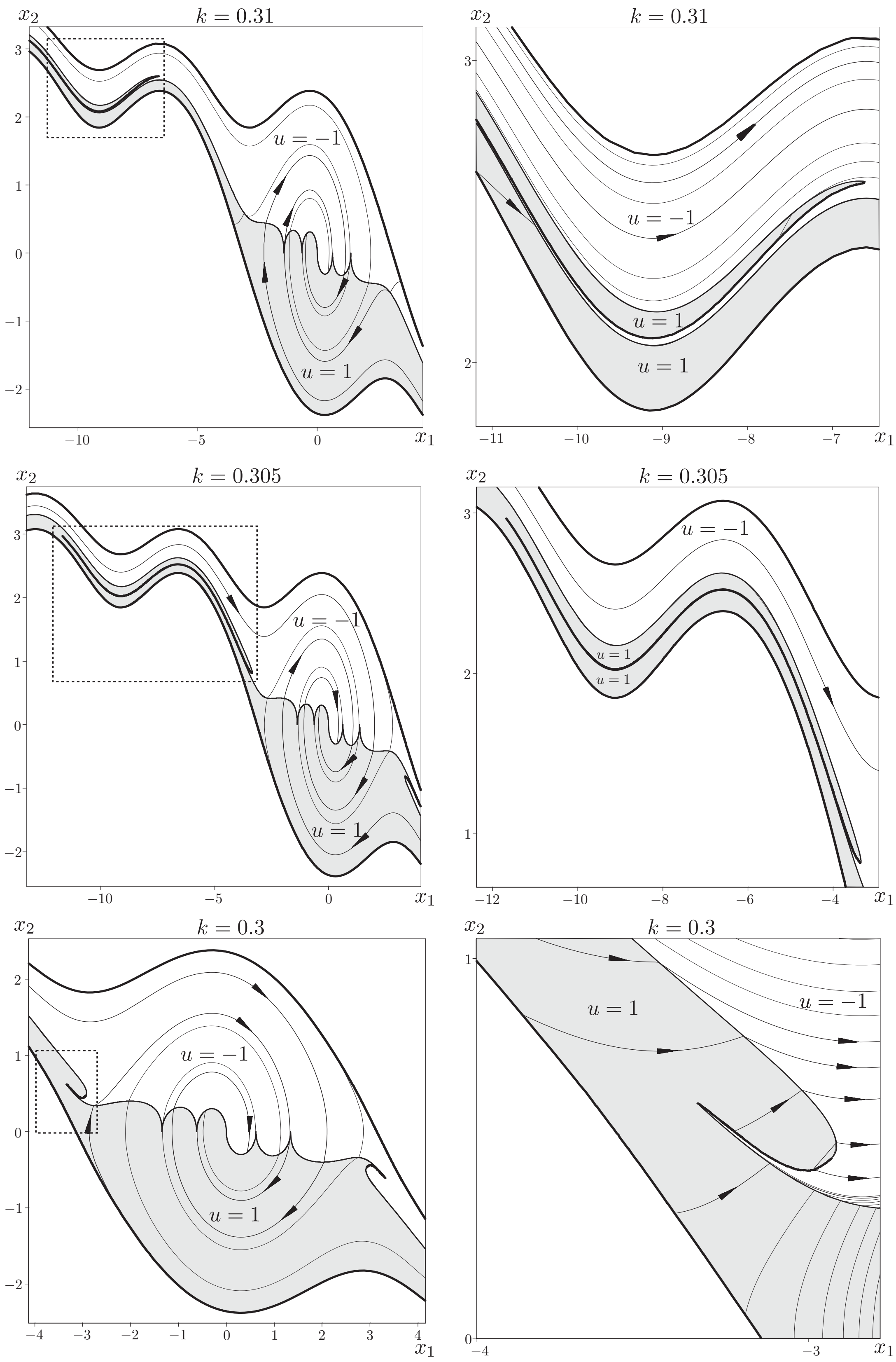

Figure 9: The time-optimal damping feedback control for $k = 0.31$, 0.305, and 0.3.

8 References

[1] Beletskii V. V.: *On optimal steering of an artificial Earth's satellite in a gravitationally stable position*. Kosm. Issl., 9 (3) (1971), 366–375.
[2] Chernousko F. L. and Reshmin S. A.: *Time-optimal swing-up feedback control of a pendulum*. Nonlinear Dynamics, 47 (1–3) (2007), 65–73.
[3] Friedland B. and Sarachik P.: *Indifference regions in optimum attitude control*. IEEE Trans. Autom. Control, 9 (2) (1964), 180–181.
[4] Garcia Almuzara J. L. and Flügge-Lots I.: *Minimum time control of a nonlinear system*. J. Differen. Equations, 4 (1) (1968), 12–39.
[5] Lee E. B. and Markus L.: *Foundations of Optimal Control Theory*. Wiley, New York, 1967.
[6] Pontryagin L. S., Boltyanskii V. G., Gamkrelidze R. V., and Mishchenko E. F.: *The Mathematical Theory of Optimal Processes*. Gordon and Breach, New York, 1986.
[7] Reshmin S. A.: *Finding the principal bifurcation value of the maximum control torque in the problem of optimal control synthesis for a pendulum*. Journal of Computer and Systems Sciences International, 47 (2) (2008), 163–178.
[8] Reshmin S. A. and Chernousko F. L.: *Time-optimal control of an inverted pendulum in the feedback form*. Journal of Computer and Systems Sciences International, 45 (3) (2006), 383–394.
[9] Reshmin S. A. and Chernousko F. L.: *Time-optimal feedback control of a nonlinear pendulum*. Journal of Computer and Systems Sciences International, 46 (1) (2007), 9–18.