QUATERNIONIC METHODS IN MATHEMATICAL PHYSICS

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Abstract. Quaternionic analysis has become an important tool in the analysis of partial differential equations and their application in mathematical physics and engineering. The main goal of my talk is to offer a quaternionic modelling and solution theory. This contribution can be applied for a wide range of classes of problems in mathematical physics under given initial value and boundary value conditions. In this short article we have to restrict our studies to one example.

1 Introduction

Quaternionic analyis and also Clifford analysis have in recent years become increasingly important tools in the analysis of partial differential equations. In particular fluid flow problems, Maxwell equations and equations in 3D-elasticity has been considered. The talk reflects the meaning and the importance of quaternionic operator methods for the treatment of boundary value problems and initial boundary value problems of stationary and non-stationary linear and some non-linear equations in fluid dynamics. We will give a survey of problems which can be successfully dealt with quaternionic methods. The scope of these problems reachs from classical Navier–Stokes equations for Newtonian fluids up to viscous fluids under the influence of temperature or field induction and problems in elasticity and electro-magnetism.

A corresponding discrete calculus exists and is worked out in ([2]). The technique is demonstrated for the stationary Navier–Stokes equation with heat conduction.In the case of initial boundary value problems a time-discretization method is used. Aside of problems which are based on Navier–Stokes equations (Benard's problem, shallow water wave equations etc.) also fluid flow problems on the sphere (forecasting equations) and through porous media (Galpern-Sobolev equations) are studied (cf. ([4])) We will give a little insight in this theory by consideration of Benard's problem. .

The essential new element in this approach is the use of a quaternionic operator calculus, which is generated by three operators: an algebraical integral operator the so called Teodorescu transform, an algebraical differential operator the so called generalized Dirac operator and an initial-value operator, which is identified with a Cauchy–Fueter operator.

2 Quaternionic Operator Calculus

2.1 Quaternions and quaternionic valued functions

Let $\mathbf{H}$ be the algebra of real quaternions and $a \in \mathbf{H}$, then $a = \sum_{k=0}^{3} \alpha_k e_k$. Further let be $e_k^2 = -e_0; e_1 e_2 = -e_2 e_1 = e_3, e_2 e_3 = -e_3 e_2 = e_1, e_3 e_1 = -e_1 e_3 = e_2$. Natural operations of addition and multiplication in $\mathbf{H}$ turn $\mathbf{H}$ to a skew-field. Quaternionic conjugation is given by

$$\overline{e_0} = e_0, \overline{e_k} = -e_k \quad (k = 1,2,3), \quad \overline{a} = a_0 - \sum_{k=1}^{3} \alpha_k e_k = a_0 - \mathbf{a}, \tag{1}$$

$$\overline{a}a = a = |a|^2_{\mathbf{R}^4} =: |a|^2_{\mathbf{H}}, \tag{2}$$

$$a^{-1} := \frac{1}{|a|^2}\overline{a}, \quad \overline{ab} = \overline{b}\,\overline{a}. \tag{3}$$

As a structure quternions were discovered by Sir R. W. Hamilton in 1843. Already 100 years earlier L. Euler used such units in his theory of kinematics. A similar multiplication rule was also found in the diary of C.F. Gauss (1823).

We denote by $\mathbf{H}(\mathbf{C})$ the set of quaternions with complex coefficients, i.e.

$$a = \sum_{k=0}^{3} \alpha_k e_k \quad (\alpha_k \in \mathbf{C}). \tag{4}$$

For $k = 0, 1, 2, 3$ we have the commutator relation $ie_k = e_k i$. Any complex quaternion a has the decomposition $a = a^1 + ia^2 \quad (a^j \in \mathbf{H})$ and therefore also the denotation $\mathbf{CH}$ can be used. We have three possible conjugations: $\overline{a}^{\mathbf{C}} := a^1 - ia^2$,, $\quad \overline{a}^{\mathbf{H}} := \overline{a}^1 + i\overline{a}^2$ and $\overline{a}^{\mathbf{CH}} := \overline{a}^1 - i\overline{a}^2$.

Let G be a bounded domain in $\mathbf{R}^3$ and $\partial G := \Gamma$. Further let $p \geq 1$ then Sobolev spaces $W_p^k(G)$ $k \in \mathbf{N}$ as well as Sobolev-Slobodetzkij spaces $W_p^k(\Gamma)$ $k = [k] + \lambda$, $\lambda \in (0,1)$ for quaternion-valued functions are componentwise defined.

2.2 Quaternionic operator trinity

Let $X = W_p^k(G), Y = W_p^{k+1}(G), Z = W_p^{k-(1/p)+1}(\Gamma); k = 0, 1, 2, ...; 1 < p < \infty$. We introduce the following linear operators:

1. Dirac operator with zero-mass:

$$(D_{ia}u)(x) := (\partial_1 e_1 + ... + \partial_3 e_3 + iae_0)u(x) : \quad Y \to X,$$

2. We consider the kernel function

$$e_a(x) := -\left(\frac{ia}{2\pi}\right)^{3/2} [f_{ia}(|x|)\omega - g_{ia}(|x|)] \quad \text{and} \tag{5}$$

$$f_{ia}(t) := t^{-1/2} K_{3/2}(iat) \quad g_{ia}(t) := t^{-1/2} K_{1/2}(iat), \tag{6}$$

where $\omega = x/|x| \in S^2$ (unit sphere in $\mathbf{R}^3$) and $K(z)$ $(z \in \mathbf{C})$ denotes MacDonald's function. The function e_a is a fundamental solution of D_{ia}. Let $u \in C(G)$. The weakly singular integral operator

$$(T_{ia}u)(x) := \int_G e_a(y-x)u(y)dy, \quad x \in G \tag{7}$$

is called the generalized Teodorescu transform. Notice that T_{ia} is a right inverse to D_{ia}. Therefore, for $u \in C(G)$ we have $(D_{ia}T_{ia}u)(x) = u(x) \quad x \in G$.

3. Let $u \in C^1(G) \cap C(\overline{G})$. Then the operator

$$(F_{ia}u)(x) := \int_\Gamma e_a(x-y)n(y)u(y)d\Gamma_y \quad x \in G \cup (\mathbf{R}^3 \setminus \overline{G}) \tag{8}$$

is called Cauchy–Fueter operator. Here $n(y)$ denotes the unit vector of the outward pointing normal at the point y. It is easy to see that $(D_{ia}F_{ia}u)(x) = 0 \quad \text{in} \quad G \cup \mathbf{R}^3 \setminus \overline{G}$. Furthermore, it holds a formula of Borel–Pompeiu type:

$$(F_{ia}u)(x) + (T_{ia}D_{ia}u)(x) = u(x) \quad \text{in} \quad G. \tag{9}$$

This formula can be extended by continuity to functions $u \in W_2^1(G)$ and their traces in $W_2^{\frac{1}{2}}(\Gamma)$, respectively. For more detail have a look in [1]. The choice of the function $a = a(t)$ depends on the problem.

2.3 Bergman–Hodge decomposition

Further, let $\overline{\partial} := \partial_0 - D$. This operator is also called Cauchy-Fueter operator. Functions of the class $(\ker\partial)(G) \cap C^1(\overline{G})$ are called ∂-holomorphic or simply holomorphic.

In ([2]) is proved: *The set* $\ker\partial(G) \cap W_p^k(G)$ *is closed in* $W_p^k(G)$ *and called the* Bergman space.

Proof. For the proof we only need the mean value formula for holomorphic fuctions and Weierstrass' theorem for sequences of holomorphic functions. The proof is similar to [2]. #

In [2] we obtained the statement: *Let* $\partial = \partial_0 + D$ *with* $D = \partial_1 e_1 + \partial_2 e_2 + \partial_3 e_3$, $G \subset \mathbf{R}^4$. *The Hilbert space* $L_2(G)$ *submits the orthogonal decomposition:*

$$L_2(G) = (\ker\partial \cap L_2)(G) \oplus \overline{\partial}\, \overset{\circ}{W}{}_2^1(G) \tag{10}$$

with respect to the inner product

$$(u,v)_2 = \int_G \overline{u(y)}v(y)dG_y. \tag{11}$$

The generalized projection of Bergman type permits an explicit representation within our calculus. It holds that

$$u \in \mathrm{im}\mathbf{Q} \quad \text{if and only if} \quad \mathrm{tr}_\Gamma T u = 0, \tag{12}$$

where

$$\mathbf{P} := F_\Gamma(tr_\Gamma TF)^{-1} tr_\Gamma T, \qquad \mathbf{Q} := I - \mathbf{P}.$$

The definition of the operator **P** is justified by the validity of the following statement:

The operator

$$\mathrm{tr}_\Gamma TF_\Gamma : \mathrm{im}\, P_\Gamma \cap W_2^{1/2}(\Gamma) \rightarrow \ \mathrm{im}\, Q_\Gamma \cap W_2^{2/3}(\Gamma) \tag{13}$$

is an isomorphism, where P_Γ and Q_Γ denote the corresponding Plemelj projections.

For more detail have a look in [1].

3 Stationary Boussinesq problem with real quaternionic methods

3.1 Boussinesq equation in quaternionic formulation

We will now prescind from the physical interpretation of the equations. Here, b_1, b_2, b_3, b_4 are positive constants, $\mathbf{u}$ denotes an unknown vector-valued function and w an unknown scalar-valued function. Our problem then reads:

$$\begin{aligned} -\Delta\mathbf{u} + b_1(\mathbf{u}\cdot\nabla)\,\mathbf{u} + b_2\nabla p + f(\mathbf{u}) - b_3(e_3)w &= F(x), & (14)\\ \mathrm{div}\,\mathbf{u} &= 0, & (15)\\ -\Delta w + b_4(\mathbf{u}\cdot\nabla)\,w &= g. & (16) \end{aligned}$$

In quaternionic notation the problem can be described as follows: Let $u = u_0 + \mathbf{u}$. Then we have to add the trivial boundary value problem $\Delta u_0 = 0$ in G and $u_0 = 0$ on Γ. Moreover we identify the scalar function $p = p(x)$ with $(p(x),0,0,0)^T$ and we put

$$M(u) := b_1(u\cdot\nabla)u - F(x) + f(u).$$

Problem (**??**)then has the formulation:

$$\begin{aligned} D^2u + M(u) + b_2\nabla p - b_3e_3w &= 0 \quad \text{in} \quad G, & (17)\\ \mathrm{Sc}\,Du &= 0 \quad \text{in} \quad G, & (18)\\ D^2w - b_4\mathrm{Sc}\,(uD)w &= g \quad \text{in} \quad G, & (19)\\ u &= 0 \quad \text{on} \quad \Gamma, & (20)\\ w &= 0 \quad \text{on} \quad \Gamma. & (21) \end{aligned}$$

3.2 Quaternionic operator-integral equations

Using Borel-Pompeiu's formula we obtain after application of the operator $T\mathbf{Q}T$ from the right

$$\begin{aligned} u &= -T\mathbf{Q}T[M(u) + b_3e_3w] - b_2T\mathbf{Q}p & (22)\\ 0 &= Sc\{\,b_1\mathbf{Q}T[M(u) + b_3e_3w] + b_2\mathbf{Q}p\} & (23)\\ w &= b_4T\mathbf{Q}TSc\,(uDw) + T\mathbf{Q}Tg. & (24) \end{aligned}$$

Notice that for $\mathrm{tr}_\Gamma T\mathbf{Q}u = 0$ the boundary conditions for u and w are fulfilled. The choice of the function $a = a(t)$ depends on the problem, in our case $a(t) = 0$, for simplicity. By the help of a general trace operator Plemelj type formulae are deduced. In a pair of Hardy type spaces a generalized potential operator is an isomorphism in the scale of (real and complex) quaternionic Sobolev-Slobodetzki spaces. With the aid of so called Bergman type projections operator representations of solutions of the corresponding problem are obtained. Numerical considerations are worked out. Convergence and error estimates could be shown.

This approach has the advandage that modelling, solution theory and numerical treatment can be studied from a unique point of view. An additional algebraical structure can be useful for simulation methods.

4 References

[1] K. Gürlebeck,K. Habetha and W. Sprössig (2008) *Holomorphic Functions in the Plane and n-dimensional Space*,Birkhäuser Basel.

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