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Modeling and Optimal Operation of Robotic Manipulators with Discrete Controls

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Abstract. A model for an industrial robot with discrete controls is presented. Transitions between different control levels are not idealized by step functions, but described more realistically by continuous transition profiles. During transition time, no control commands are accepted by the respective control. This leads to a control problem with reduced dimension of the control vector during response times. A Maximum Principle based approach is used to calculate optimal trajectories with high accuracy. The original optimal control problem is embedded into a continuous problem of higher dimension. By this, optimality conditions are not only evaluated at discrete times, but the system dynamics and sensitivity properties on the full solution interval are coupled to the switching points of the system. The extended problem of optimal control is transformed into a multi-point boundary value problem on a fixed time grid and of piecewise varying dimension. By recursive modeling techniques, the formulation of the boundary value problem is automated to a great extent. An advanced multiple shooting method provides significantly improved stability of the numerical solution process. Time-optimal three-dimensional trajectories are calculated for a three-link example robot. With discrete controls properly determined, a similar performance of the robotic manipulator is achieved as with continuous controls. However, the solution structure is more complicated than in the continuous case, especially if state constraints are active.

1 Introduction

The use of hydraulic or pneumatic actuators in industrial robots allows to generate high specific forces in a compact design. A further simplification of the manipulator design becomes possible by the restriction to only discrete levels for some or all of the control variables (= torques) involved. This is the case, if e.g. magnetic valves are installed. It is no longer necessary to generate continuous pressure profiles with high accuracy, only exact switching times have to be realized. The price to be paid for a simpler electromechanical design might be a loss in overall performance.

It is the aim of this article to set up a mathematical model for such an industrial robot and carefully investigate the effects of discrete controls compared with the continuous case.

2 System modeling

2.1 Manipulator model

The three-dimensional rigid body model of the manipulator reflects the main kinematical and dynamical properties. Non-rigid body effects like joint friction or motor models are neglected in this paper, but fit seamlessly into the framework presented here. The example manipulator is modeled as a chain of n rigid links connected by revolute joints. Joint angles are denoted by θ_i, the corresponding actuator torques by T_i, $i = 1,\ldots,n$. The links are numbered starting from the immobile base of the manipulator (link 0).

Each revolute joint i is controlled via the joint angle θ_i. The joint variables $\Theta := (\theta_1,\ldots,\theta_n)^T \in \mathbf{R}^n$ form a complete set of minimum coordinates.

The equations of motion (i.e. the dynamic equations) of an n-link rigid body manipulator in the joint space are calculated by a modified version [9] of the recursive Newton-Euler algorithm (cf. e.g. [10, 15, 17, 18]) in an implicit form (inverse dynamics):

First, link velocities and accelerations are computed from link 1 to link n (outward recursion) applying Newton-Euler formulae to each link. In a second step, joint forces and torques are computed from link n to link 1 (inward recursion). For each set of input values $(\Theta,\dot{\Theta},\ddot{\Theta})$, the output of the recursions are $M(\Theta)$, $h(\Theta,\dot{\Theta})$ and a set of torques

$$T = T(\Theta,\dot{\Theta},\ddot{\Theta}) = M(\Theta)\ddot{\Theta} + h(\Theta,\dot{\Theta}) \,, \quad T := (T_1,\ldots,T_n)^T \in \mathbf{R}^n \,. \tag{1}$$

$M(\Theta) \in \mathbf{R}^{n\times n}$ is the manipulator joint space inertia matrix. It is both symmetric and positive definite (and therefore always invertible) and is composed of those terms which multiply $\ddot{\Theta}$. The function $h(\Theta,\dot{\Theta}) \in \mathbf{R}^n$ contains all terms which depend on the joint angles or the joint velocities, as this is the case with terms caused by gravitational, centrifugal and Coriolis forces. A further subdivision of the effects modeled in $h(\Theta,\dot{\Theta})$ is not necessary for the numerical treatment [7].

For optimal control applications, efficient access to accurate and detailed derivative information is crucial. The first derivatives of the rigid body system

$$\frac{\partial M(\Theta)}{\partial \theta_j}, \frac{\partial h(\Theta,\dot{\Theta})}{\partial \theta_j}, \frac{\partial h(\Theta,\dot{\Theta})}{\partial \dot{\theta}_j}$$

and the corresponding derivatives of the torques

$$\begin{aligned} \frac{\partial T}{\partial \theta_j} &= \frac{\partial T(\Theta,\dot{\Theta},\ddot{\Theta})}{\partial \theta_j} = \frac{\partial M(\Theta)}{\partial \theta_j}\ddot{\Theta} + \frac{\partial h(\Theta,\dot{\Theta})}{\partial \theta_j}, \quad j = 1,\ldots,n \\ \frac{\partial T}{\partial \dot{\theta}_j} &= \frac{\partial T(\Theta,\dot{\Theta},\ddot{\Theta})}{\partial \dot{\theta}_j} = \frac{\partial h(\Theta,\dot{\Theta})}{\partial \dot{\theta}_j}, \quad j = 1,\ldots,n \end{aligned}$$

are obtained by differentiating the Newton-Euler recursion with respect to θ_j, $\dot{\theta}_j$. For each joint j, this differentiation process again results in an outward and an inward recursion. Quantities previously calculated by the original Newton-Euler algorithm are reused. Higher-order derivatives are calculated analogously [9].

With the vector of the state variables $x(t) := (\Theta,\dot{\Theta})^T \in \mathbf{R}^{2n}$ and the vector of the linear controls $u := T \in \mathbf{R}^n$, the equations of motion (1) may be rearranged in a normalized form

$$\dot{x}(t) = \left(\begin{array}{c} \dot{\Theta}(t) \\ M^{-1}(\Theta(t))\left(u(t) - h(\Theta(t),\dot{\Theta}(t))\right) \end{array} \right). \tag{2}$$

The explicit form (2) is especially suited for numerical calculations. $M^{-1}(\Theta(t))$ is never computed explicitly, but $\ddot{\Theta}(t)$ solves the linear system

$$M(\Theta(t))\ddot{\Theta}(t) = u(t) - h(\Theta(t),\dot{\Theta}(t)).$$

2.2 Control and state variable inequality constraints

The basic control inequality constraints are of interval type

$$T_i(t) \in [T_{i,min}, T_{i,max}] \quad \forall\, t \in [\tau_0,\tau_f], \quad i = 1,\ldots n. \tag{3}$$

τ_0 is the initial time and τ_f the terminal time for the manipulator motion.

In case of discrete controls, additional and severe restrictions are imposed on the control variables T_i:

$$T_i \in \mathscr{T}_i := \{T_{i,1},\ldots,T_{i,\kappa_i}\}, \ i = 1,\ldots,n, \ 3 \le \kappa_i \le 10 \quad \wedge \quad T_{i,min} =: T_{i,1} < T_{i,2} < \ldots < T_{i,\kappa_i} := T_{i,max} \tag{4}$$

with κ_i set in advance. Only a finite number of discrete levels of torque is allowed for control. Between the different levels switching is possible. The set of possible controls is no longer convex.

In reality, an ideal step control cannot be realized. The manipulator's motion is rather characterized by a sequence of directly controlled and uncontrolled (or indirectly controlled) arcs.

On directly controlled arcs, $T_i \in \mathscr{T}_i \ \forall i \in \{1,\ldots,n\}$. When a command is given at an intermediate time $\tau_s \in]\tau_0,\tau_f[$ to change the control level of a single T_j, $j \in \{1,\ldots,n\}$ from $T_{j,s-}$ to $T_{j,s+}$ with $T_{j,s-}, T_{j,s+} \in \mathscr{T}_j \ \wedge \ T_{j,s-} \neq T_{j,s+}$, this change cannot be performed immediately. The torque function is modeled to follow a given profile function $S_j \in \mathscr{C}^2([0,\Delta\tau_j] \times [T_{j,min},T_{j,max}]^2, \mathbf{R})$ during a characteristic response time $\Delta\tau_j$

$$T_j(t) = S_j(t-\tau_s, T_{j,s-}, T_{j,s+}) \quad \forall\, t \in [\tau_s, \tau_s + \Delta\tau_j] \cap [\tau_0,\tau_f]. \tag{5}$$

The function S_j can be measured, but not altered during operation. It depends on the the control level $T_{j,s-}$ prior to τ_s and the selected level $T_{j,s+}$ afterwards. Due to over-/undershooting, the basic control inequality constraints $[T_{j,min},T_{j,max}]$ cannot be applied to $T_j(t)$, $t \in [\tau_s, \tau_s + \Delta\tau_j]$.

During the time interval $[\tau_s, \tau_s + \Delta\tau_j]$, no control command can be given to the control variable T_j; the set of available controls is reduced by T_j and so is the dimension of the control vector.

Minimum requirements for a valid profile model function S_j are

$$S_j(0,T_{j,s-},T_{j,s+}) = T_{j,s-}, \quad S_j(\Delta\tau_j,T_{j,s-},T_{j,s+}) = T_{j,s+}.$$

A higher order of differentiability of the control function $T_j(t)$ can be achieved by the additional conditions

$$\begin{aligned} \frac{\partial}{\partial \xi} S_j(\xi,T_{j,s-},T_{j,s+})\Big|_{\xi=0} &= 0, & \frac{\partial}{\partial \xi} S_j(\xi,T_{j,s-},T_{j,s+})\Big|_{\xi=\Delta\tau_j} &= 0, \\ \frac{\partial^2}{\partial \xi^2} S_j(\xi,T_{j,s-},T_{j,s+})\Big|_{\xi=0} &= 0, & \frac{\partial^2}{\partial \xi^2} S_j(\xi,T_{j,s-},T_{j,s+})\Big|_{\xi=\Delta\tau_j} &= 0, \\ & \ldots \end{aligned}$$

which can be realized e.g. by a proper choice of the spline functions modeling the profile function S_j.

In addition, multiple state inequality constraints of the type

$$C(x) \leq 0 \quad \text{with} \quad C \in \mathscr{C}^{\mu}(\mathbf{R}^{2n}, \mathbf{R}), \ \ \mu \geq 2,$$

have to be fulfilled.

2.3 Performance index and boundary conditions

Without loss of generality, an objective function of Mayer type is chosen

$$I := \Phi\left(x(\tau_0^+), x(\tau_f^-)\right) \quad \longrightarrow \quad \min!$$

For point-to-point trajectories, the model is completed by boundary conditions, e.g. joint angles and angular velocities at initial time τ_0 and at final time τ_F are prescribed:

$$\Theta(\tau_0) = \Theta_0\,, \quad \dot{\Theta}(\tau_0) = \dot{\Theta}_0\,, \quad \Theta(\tau_F) = \Theta_F\,, \quad \dot{\Theta}(\tau_F) = \dot{\Theta}_F\,. \tag{6}$$

3 Optimal control of the manipulator

3.1 Demonstrator problem in infinite-dimensional control spaces

To avoid merely technical discussions and to concentrate on the mathematical core of the matter, a demonstrator problem with reduced complexity is analyzed in detail.

Mathematically, the demonstrator problem can be stated as follows:

Find a state function $\quad x : [\tau_0, \tau_f] \longrightarrow \mathbf{R}^{2n}$
and a control function $\quad u : [\tau_0, \tau_f] \longrightarrow \mathbf{R}^{n}$,

which minimize an objective function of Mayer type

$$\Phi\left(x(\tau_0^+), x(\tau_f^-)\right) \tag{7}$$

subject to

$$\left.\begin{array}{rcl} \dot{x} & = & f(x,u)\,, \\ 0 & = & x(\tau_0) - x_0\,, \\ u_k(t) & \in & [u_k^-, u_k^+], \ \ k = 1, \ldots, n\,. \end{array}\right\} \tag{8}$$

x is assumed to be an element of space $\mathscr{W}^{1,\infty}([\tau_0, \tau_f], \mathbf{R}^{2n})$ of the uniformly Lipschitz-continuous functions under the norm $\|x\|_{1,\infty} := \|x(\tau_0)\|_2 + \|\dot{x}\|_\infty$; $\|\cdot\|_2$ denotes the Euclidean norm and $\|\dot{x}\|_\infty := \text{ess sup}\{\|\dot{x}(t)\|_2 \big| t \in [\tau_0, \tau_f]\}$. u is assumed to be an element of space $L^\infty([\tau_0, \tau_f], \mathbf{R}^p)$ of the bounded functions under the norm $\|u\|_\infty$.

$\left(\mathscr{W}^{1,\infty}([\tau_0, \tau_f], \mathbf{R}^{2n}), \|\cdot\|_{1,\infty}\right)$ and $\left(L^\infty([\tau_0, \tau_f], \mathbf{R}^p), \|\cdot\|_\infty\right)$ are Banach spaces.

τ_0 is the initial time and τ_f the final time. Let be $\bar{U}_1 \subseteq \mathbf{R}^{2n}$ and $\bar{U}_2 \subseteq \mathbf{R}^n$ open sets such that the optimal solution $(x^\star, u^\star) \in \bar{U} := \bar{U}_1 \times \bar{U}_2 \ \ \forall\, t \in [\tau_0, \tau_f]$. By construction, $f \in \mathscr{C}^\infty(\bar{U}, \mathbf{R}^{2n})$.

3.2 Modified demonstrator problem in finite-dimensional control spaces

This problem is a modification of (7,8). Now a special control history is assumed for the optimal solution: All controls except one are constant in $[\tau_0, \tau_f]$. The remaining control has exactly one switching point $\tau_s \in]\tau_0, \tau_f - \Delta\tau[$, at which the control starts a transition between two constant control levels.

Problem definition. The modified demonstrator problem in finite-dimensional control spaces reads as follows:

Find a state function $\quad x : [\tau_0, \tau_f] \longrightarrow \mathbf{R}^{2n}$
and $n+1$ control parameters $\quad \{u_{0,1}, \ldots, u_{0,n}, u_{1,n}\} \subset \mathbf{R}$,

which minimize an objective function of Mayer type

$$\Phi\left(x(\tau_0^+), x(\tau_f^-)\right) \tag{9}$$

subject to

$$\left.\begin{array}{rcl} \dot{x} & = & f(x,u) \quad \text{with} \\ & & u(t) = \left\{ \begin{array}{ll} (u_{0,1},\ldots,u_{0,n-1},u_{0,n})^T & \forall\, t \in [\tau_0,\tau_s[\,, \\ (u_{0,1},\ldots,u_{0,n-1},S(t-\tau_s,u_{0,n},u_{1,n}))^T & \forall\, t \in [\tau_s,\tau_s+\Delta\tau[\,, \\ (u_{0,1},\ldots,u_{0,n-1},u_{1,n})^T & \forall\, t \in [\tau_s+\Delta\tau,\tau_f[\,, \end{array}\right. \\ 0 & = & x(\tau_0)-x_0\,, \\ 0 & = & S(0^+,u_{0,n},u_{1,n})-u_{0,n}\,, \\ 0 & = & S(\Delta\tau^-,u_{0,n},u_{1,n})-u_{1,n}\,, \\ & & u_{0,n},u_{1,n} \in [u_n^-,u_n^+], \quad u_{0,k} \in [u_k^-,u_k^+], \;\; k=1,\ldots,n-1\,. \end{array}\right\} \tag{10}$$

x is assumed to be an element of space $\mathscr{C}^1([\tau_0,\tau_f],\mathbf{R}^{2n})$ of the continuously differentiable functions, u by construction is an element of space $\mathscr{C}^0([\tau_0,\tau_f],\mathbf{R}^n)$ of the continuous functions.
$\|\varphi\|_{\mathscr{C}^k([\tau_0,\tau_f],\mathbf{R}^n)} := \sum_{j=1}^n \sum_{\nu=0}^k \max_{t\in[\tau_0,\tau_f]} \left|\varphi_j^{(\nu)}(t)\right|$ defines a norm $\forall\, \varphi = (\varphi_1,\ldots,\varphi_n)^T \in \mathscr{C}^k([\tau_0,\tau_f],\mathbf{R}^n)$.

$\left(\mathscr{C}^1([\tau_0,\tau_f],\mathbf{R}^{2n}), \|\cdot\|_{\mathscr{C}^1([\tau_0,\tau_f],\mathbf{R}^{2n})}\right)$ and $\left(\mathscr{C}^0([\tau_0,\tau_f],\mathbf{R}^n), \|\cdot\|_{\mathscr{C}^0([\tau_0,\tau_f],\mathbf{R}^n)}\right)$ are Banach spaces.

$\Delta\tau$ is the characteristic response time and τ_s as well as $\tau_s+\Delta\tau$ are intermediate times with interior point conditions. Let the open sets $\bar{U}_1 \subseteq \mathbf{R}^{2n}$ and $\bar{U}_2 \subseteq \mathbf{R}^n$ be chosen such that also for the optimal solution $(x^{\star\star},u^{\star\star})$ of the modified problem in finite-dimensional control spaces holds: $(x^{\star\star},u^{\star\star}) \in \bar{U} := \bar{U}_1 \times \bar{U}_2 \;\; \forall\, t \in [\tau_0,\tau_f]$. By construction, $f \in \mathscr{C}^\infty(\bar{U},\mathbf{R}^{2n})$ (from (2)) and $S \in \mathscr{C}^2([0,\Delta\tau]\times[u_n^-,u_n^+]\times[u_n^-,u_n^+],\mathbf{R})$ (from spline interpolation). As usual we define $x(\tau_j^\pm) := \lim_{\varepsilon\to 0,\varepsilon>0} x(\tau_j \pm \varepsilon)$.

Because the profile function S explicitly depends on t, the demonstrator problem is non-autonomous. It can be transformed into an autonomous problems by defining the time t as a new and additional state variable.

Control problems with free final time τ_f can be transformed into problems with fixed final time by introducing the new independent variable s with $s := t/\tau_f$, $s \in [0,1]$, and the new state variable τ_f with $\dot{\tau}_f = 0$ [12]. Bolza and Lagrange problems can be converted into Mayer problems of type (9) by standard transformations [3, 6].

Embedding into a continuous problem. Because of the additional conditions on $u(t)$ in (10), the complete control structure is fixed by choosing the interior time τ_s and the $n+1$ control parameters $u_{0,1},\ldots,u_{0,n},u_{1,n}$. The control space has a finite dimension of $n+2$ in contrast to the infinite-dimensional case, in which n control *functions* $u_k(t) \in \mathbf{R}$, $k=1,\ldots,n$, have to be determined $\forall\, t \in [\tau_0,\tau_f]$.

With the constraints $u_{0,n},u_{1,n} \in [u_n^-,u_n^+]$, $u_{0,k} \in [u_k^-,u_k^+]$, $k=1,\ldots,n-1$, from (10), the set of admissible control parameters is convex.

To obtain an unconstrained problem, the control functions are transformed

$$u_k \to w_k \quad \text{with} \quad u_k(t) =: u_k^- + (u_k^+ - u_k^-)\sin^2 w_k(t) \quad \forall\, t \in [\tau_0,\tau_f]\,, \quad k=1,\ldots,n-1\,, \tag{11}$$

$$u_n(t) =: u_n^- + (u_n^+ - u_n^-)\sin^2 w_n(t) \quad \forall\, t \in [\tau_0,\tau_s[\cup[\tau_s+\Delta\tau,\tau_f]\,. \tag{12}$$

In the next step, the derivatives of the control functions $w_k(t)$ are chosen as new controls, thus transforming the original controls $w_k(t)$ into state variables. The original state and the new control variables are in intervals combined into a new state vector

$$z := \begin{pmatrix} x \\ w \end{pmatrix} : [\tau_0,\tau_s[\,\cup\,[\tau_s+\Delta\tau,\tau_f] \to \mathbf{R}^{2n}\times\mathbf{R}^n \quad \wedge \quad z := \begin{pmatrix} x \\ \tilde{w} \\ u_n \end{pmatrix} : [\tau_s,\tau_s+\Delta\tau[\to \mathbf{R}^{2n}\times\mathbf{R}^n$$

with $\tilde{w} := (w_1,\ldots,w_{n-1})^T$ and $\tilde{u} := (u_1,\ldots,u_{n-1})^T$. The extended right hand side of the system of differential equations $\bar{f} : \mathbf{R}^{2n}\times\mathbf{R}^n\times\mathbf{R} \to \mathbf{R}^{2n}\times\mathbf{R}^n$ is defined by

$$\bar{f}(z(t),t) := \left\{ \begin{array}{ll} \begin{pmatrix} f(x(t),u(w(t))) \\ 0^n \end{pmatrix} & \forall\, t \in [\tau_0,\tau_s[\cup[\tau_s+\Delta\tau,\tau_f]\,, \\ \begin{pmatrix} f(x(t),u(w(t))) \\ 0^{n-1} \\ \dfrac{\partial}{\partial t} S\left(t-\tau_s, u_n(w_n(\tau_s^-)), u_n(w_n((\tau_s+\Delta\tau)^+))\right) \end{pmatrix} & \forall\, t \in [\tau_s,\tau_s+\Delta\tau[\end{array}\right. \tag{13}$$

with the abbreviation $0^m := (0,\dots,0)^T \in \mathbf{R}^m$, $m \in \mathbb{N}$. If all the original controls are from finite-dimensional spaces, no controls are left after the transformation and the optimal control problem is converted into a special problem of the calculus of variations, which is controlled only by the proper choice of the interior point conditions. The case of a mixture of finite- and infinite-dimensional controls is treated analogously.

To get rid of the non-standard expressions $w_n(\tau_s^-)$ and $w_n((\tau_s+\Delta\tau)^+)$ in (13), two new variables $A(t)$, $B(t)$ are defined on $[\tau_s, \tau_s+\Delta\tau[$ by the differential equations $\dot{A}=0$, $\dot{B}=0$ and the interior point conditions $A(\tau_s^+) = u_n(w_n(\tau_s^-))$, $B((\tau_s+\Delta\tau)^-) = u_n(w_n((\tau_s+\Delta\tau)^+))$.

Together with the interior point conditions from (10)

$$\begin{aligned}
0 &= x(\tau_0) - x_0\,,\\
0 &= S(0^+, A(\tau_s^+), B((\tau_s+\Delta\tau)^-)) - u_n(w_n(\tau_s^-))\,,\\
0 &= S(\Delta\tau^-, A(\tau_s^+), B((\tau_s+\Delta\tau)^-)) - u_n(w_n((\tau_s+\Delta\tau)^+))\,,
\end{aligned}$$

and the continuity conditions for x and $\tilde{w}$, an equivalent continuous optimal control problem is completely defined.

Transformation into a boundary value problem. The optimal control problem is transformed in a well-known manner (see e.g. [3]) into a multi-point boundary value problem for a system of ordinary differential equations (MPBVP). For the solution of MPBVP, very efficient numerical algorithms exist [5]. Here the transformation into an autonomous two-point boundary problem is preferred to avoid case differentiations in the analytical treatment and to apply the minimum principle directly.

The state variables are redefined in intervals

$$\begin{array}{rrr}
x|_{[\tau_0,\tau_s[} \to y_1\,, & x|_{[\tau_s,\tau_s+\Delta\tau[} \to y_2\,, & x|_{[\tau_s+\Delta\tau,\tau_f]} \to y_3\,,\\
\tilde{w}|_{[\tau_0,\tau_s[} \to y_4\,, & \tilde{w}|_{[\tau_s,\tau_s+\Delta\tau[} \to y_5\,, & \tilde{w}|_{[\tau_s+\Delta\tau,\tau_f]} \to y_6\,,\\
w_n|_{[\tau_0,\tau_s[} \to y_7\,, & u_n|_{[\tau_s,\tau_s+\Delta\tau[} \to y_8\,, & w_n|_{[\tau_s+\Delta\tau,\tau_f]} \to y_9
\end{array}$$

and the time subintervals are transformed to the fixed time interval $\xi \in [0,1]$ by

$$\begin{aligned}
\xi_1 &:= \frac{t-\tau_0}{\tau_s-\tau_0} \in [0,1[\text{ for } t \in [\tau_0,\tau_s[\quad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t} = \frac{1}{\tau_s-\tau_0}\frac{\mathrm{d}}{\mathrm{d}\xi_1}\,,\\
\xi_2 &:= \frac{t-\tau_s}{\Delta\tau} \in [0,1[\text{ for } t \in [\tau_s,\tau_s+\Delta\tau[\quad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t} = \frac{1}{\Delta\tau}\frac{\mathrm{d}}{\mathrm{d}\xi_2}\,,\\
\xi_3 &:= \frac{t-\tau_s-\Delta\tau}{\tau_f-\tau_s-\Delta\tau} \in [0,1] \text{ for } t \in [\tau_s+\Delta\tau,\tau_f] \quad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t} = \frac{1}{\tau_f-\tau_s-\Delta\tau}\frac{\mathrm{d}}{\mathrm{d}\xi_3}\,,\\
\xi_i &\to \xi \quad \text{for} \quad i=1,2,3\,.
\end{aligned}$$

This yields the equivalent optimal control problem with the cost function

$$I = \Phi(y_1(0), y_3(1))$$

and the system of differential equations and boundary conditions

$$\left.\begin{array}{rclrcl}
\dot{y}_1 &=& \dfrac{\mathrm{d}}{\mathrm{d}\xi}y_1 = (\tau_s-\tau_0)f(y_1,\tilde{u}(y_4),u_n(y_7))\,, & y_1(0) &=& x_0\,,\\
\dot{y}_2 &=& \Delta\tau f(y_2,\tilde{u}(y_5),y_8)\,, & y_1(1) &=& y_2(0)\,,\\
\dot{y}_3 &=& (\tau_f-\tau_s-\Delta\tau)f(y_3,\tilde{u}(y_6),u_n(y_9))\,, & y_2(1) &=& y_3(0)\,,\\
\dot{y}_4 &=& 0\,, & & &\\
\dot{y}_5 &=& 0\,, & y_4(1) &=& y_5(0)\,,\\
\dot{y}_6 &=& 0\,, & y_5(1) &=& y_6(0)\,,\\
\dot{y}_7 &=& 0\,, & & &\\
\dot{y}_8 &=& \Delta\tau\dfrac{\partial}{\partial t}S(\Delta\tau\cdot\rho,A,B)\,, & u_n(y_7(1)) &=& y_8(0)\,,\\
\dot{y}_9 &=& 0\,, & y_8(1) &=& u_n(y_9(0))\,,\\
\dot{A} &=& 0\,, & A(0) &=& u_n(y_7(1))\,,\\
\dot{B} &=& 0\,, & B(1) &=& u_n(y_9(0))\,,\\
\dot{\tau}_s &=& 0\,, & & &\\
\dot{\rho} &=& 1\,, & \rho(0) &=& 0\,.
\end{array}\right\} \qquad (14)$$

No interior point conditions occur in this system.

With $\bar{z}(t) := \left(y_1^T, y_2^T, \ldots, y_9, A, B, \tau_s, \rho\right)^T \in \mathbf{R}^{9n+4}$, the Hamiltonian has the following form

$$\begin{aligned}
H(\bar{z},\lambda) &:= \lambda^T \dot{\bar{z}} \\
&= \lambda_{y_1}^T \cdot (\tau_s - \tau_0) f(y_1, \tilde{u}(y_4), u_n(y_7)) + \lambda_{y_2}^T \cdot \Delta\tau f(y_2, \tilde{u}(y_5), y_8) \\
&+ \lambda_{y_3}^T \cdot (\tau_f - \tau_s - \Delta\tau) f(y_3, \tilde{u}(y_6), u_n(y_9)) + \lambda_{y_8} \Delta\tau \frac{\partial}{\partial t} S(\Delta\tau \cdot \rho, A, B) + \lambda_\rho \,.
\end{aligned} \tag{15}$$

The subscript of each adjoint variable λ refers to the respective state variable.

The boundary and interior point conditions are coupled to the objective function Φ by Lagrangian multipliers $\nu_1, \nu_2, \nu_3 \in \mathbf{R}^{2n}$, $\nu_4, \nu_5 \in \mathbf{R}^{n-1}$, $\nu_6, \ldots, \nu_{10} \in \mathbf{R}$

$$\begin{aligned}
g &:= \Phi(y_1(0), y_3(1)) \\
&+ \nu_1^T (y_1(0) - x_0) + \nu_2^T (y_1(1) - y_2(0)) + \nu_3^T (y_2(1) - y_3(0)) \\
&+ \nu_4^T (y_4(1) - y_5(0)) + \nu_5^T (y_5(1) - y_6(0)) + \nu_6 (u_n(y_7(1)) - y_8(0))) \\
&+ \nu_7 (y_8(1) - u_n(y_9(0))) + \nu_8 (A(0) - u_n(y_7(1))) + \nu_9 (B(1) - u_n(y_9(0))) + \nu_{10} \rho(0) \,.
\end{aligned} \tag{16}$$

The differential equations for the adjoint variables λ are obtained from $\dot{\lambda} = -H_{\bar{z}}(\bar{z},\lambda)$ and read as

$$\left.\begin{aligned}
\dot{\lambda}_{y_1} &= -\lambda_{y_1}^T (\tau_s - \tau_0) f_{y_1}(y_1, \tilde{u}(y_4), u_n(y_7)) \\
\dot{\lambda}_{y_2} &= -\lambda_{y_2}^T \Delta\tau f_{y_2}(y_2, \tilde{u}(y_5), y_8) \\
\dot{\lambda}_{y_3} &= -\lambda_{y_3}^T (\tau_f - \tau_s - \Delta\tau) f_{y_3}(y_3, \tilde{u}(y_6), u_n(y_9)) \\
\dot{\lambda}_{y_4} &= -\lambda_{y_1}^T 2(\tau_s - \tau_0) f_{\tilde{u}(y_4)}(y_1, \tilde{u}(y_4), u_n(y_7))(y_4^+ - y_4^-) \sin y_4 \cos y_4 \\
\dot{\lambda}_{y_5} &= -\lambda_{y_2}^T 2\Delta\tau f_{\tilde{u}(y_5)}(y_2, \tilde{u}(y_5), y_8)(y_5^+ - y_5^-) \sin y_5 \cos y_5 \\
\dot{\lambda}_{y_6} &= -\lambda_{y_3}^T 2(\tau_f - \tau_s - \Delta\tau) f_{\tilde{u}(y_6)}(y_3, \tilde{u}(y_6), u_n(y_9))(y_6^+ - y_6^-) \sin y_6 \cos y_6 \\
\dot{\lambda}_{y_7} &= -\lambda_{y_1}^T 2(\tau_s - \tau_0) f_{u_n(y_7)}(y_1, \tilde{u}(y_4), u_n(y_7))(u_n^+ - u_n^-) \sin y_7 \cos y_7 \\
\dot{\lambda}_{y_8} &= -\lambda_{y_2}^T \Delta\tau f_{y_8}(y_2, \tilde{u}(y_5), y_8) \\
\dot{\lambda}_{y_9} &= -\lambda_{y_3}^T 2(\tau_f - \tau_s - \Delta\tau) f_{u_n(y_9)}(y_3, \tilde{u}(y_6), u_n(y_9))(u_n^+ - u_n^-) \sin y_9 \cos y_9 \\
\dot{\lambda}_A &= -\lambda_{y_8} \Delta\tau \frac{\partial^2}{\partial t \partial A} S(\Delta\tau \cdot \rho, A, B) \\
\dot{\lambda}_B &= -\lambda_{y_8} \Delta\tau \frac{\partial^2}{\partial t \partial B} S(\Delta\tau \cdot \rho, A, B) \\
\dot{\lambda}_{\tau_s} &= -\lambda_{y_1}^T \cdot f(y_1, \tilde{u}(y_4), u_n(y_7)) + \lambda_{y_3}^T \cdot f(y_3, \tilde{u}(y_6), u_n(y_9)) \\
\dot{\lambda}_\rho &= -\lambda_{y_8} \Delta\tau^2 \frac{\partial^2}{\partial t \partial \sigma} S(\sigma, A, B)|_{\sigma = \Delta\tau \cdot \rho}
\end{aligned}\right\} \tag{17}$$

Eq. (17) is the shortened form of e.g.

$$\dot{\lambda}_{y_4,k} = -2(\tau_s - \tau_0) \left(\lambda_{y_1,1}, \ldots, \lambda_{y_1,2n}\right) \begin{pmatrix} \dfrac{\partial f_1}{\partial u_k} \\ \vdots \\ \dfrac{\partial f_{2n}}{\partial u_k} \end{pmatrix} (y_1, u_1, \ldots u_{n-1}, u_n(y_9))|_{\tilde{u} = \tilde{u}(y_4)} \cdot$$
$$\cdot (u_k^+ - u_k^-) \sin(y_{4,k}) \cos(y_{4,k}), \quad k = 1, \ldots, n-1 \,.$$

In addition, the missing boundary conditions are obtained from the first variation of the extended objective function

$$\tilde{I} := g + \int_0^1 \left(H(\bar{z},\lambda) - \lambda^T \dot{\bar{z}}\right) \mathrm{d}t \,.$$

The variation yields the additional conditions

$$\left.\begin{aligned}
\lambda_{y_4}(0) &= 0 & \lambda_{y_7}(0) &= 0 \\
\lambda_{\tau_s}(0) &= 0 & \lambda_B(0) &= 0 \\
\lambda_{y_6}(1) &= 0 & \lambda_{y_9}(1) &= 0 \\
\lambda_{\tau_s}(1) &= 0 & \lambda_A(1) &= 0 \\
\lambda_\rho(1) &= 0 & & \\
\lambda_{y_2}(0) - \lambda_{y_1}(1) &= 0 & \lambda_{y_3}(0) - \lambda_{y_2}(1) &= 0 \\
\lambda_{y_5}(0) - \lambda_{y_4}(1) &= 0 & \lambda_{y_6}(0) - \lambda_{y_5}(1) &= 0 \\
\frac{\partial \Phi}{\partial y_3(1)} - \lambda_{y_3}(1) &= 0 & & \\
(u_n^+ - u_n^-)(\lambda_{y_8}(1) + \lambda_B(1)) \sin(2y_9(0)) - \lambda_{y_9}(0) &= 0 & & \\
(u_n^+ - u_n^-)(\lambda_{y_8}(0) + \lambda_A(0)) \sin(2y_7(1)) - \lambda_{y_7}(1) &= 0 & &
\end{aligned}\right\} \tag{18}$$

(18) together with (14, 17) complete the boundary value problem. The backward transformation to a MPBVP is straightforward.

Remarks. The embedding approach developed above is especially attractive in case of a mixture of finite- and infinite-dimensional controls.

Modifying and extending the basic ideas of a proof outlined in [13, 6, 5] for piecewise constant controls, the following property can be shown: The embedding approach is comparable to the direct application of algorithms for nonlinear optimization to (9,10) regarding the computational effort, if the derivatives in the direct approach are calculated with high accuracy. The exact proof is rather lengthy. Compared with nonlinear optimization, pointwise relations between the adjoint variables are derived here in addition and the system dynamics and sensitivity properties on the total time interval are coupled to the switching points of the system via the adjoint variables. All this information is used to eliminate suboptimal solutions.

3.3 Active constraints in case of finite-dimensional control spaces

The demonstrator system is also used to investigate the properties of solutions subject to state inequality constraints. The results can be directly generalized.

Without loss of generality, a state constraint $C(x) \leq 0$ is assumed to be active on the interval $[\tau_1, \tau_2] \subset [\tau_0, \tau_s[$, i.e.

$$C(x) = 0 \quad \forall\, t \in [\tau_1, \tau_2]\,. \tag{19}$$

Because of (10), the locally unique optimal solution

$$x(t; \tau_0, x(\tau_0^+), u_{01}, \ldots, u_{0n})$$

of (9, 10) on $[\tau_0, \tau_s[$ depends on the initial values $(\tau_0, x(\tau_0^+))$ and the n parameters $u_{01}, \ldots, u_{0n}$ that characterize the optimal constant controls. Insertion into (19) yields a system of nonlinear equations

$$C(x(t; \tau_0, x(\tau_0^+), u_{01}, \ldots, u_{0n})) = 0 \quad \forall\, t \in [\tau_1, \tau_2]\,. \tag{20}$$

If (20) is evaluated at more than n time instances t and especially at infinitely many times t within $[\tau_1, \tau_2]$, in general the resulting nonlinear system for the calculation of the n control parameters is overdetermined and cannot be solved exactly. This contradicts the original assumption. For the constraint $C(x) \leq 0$, therefore only contact points τ_c are possible. Because of the differentiability assumptions on (9, 10) and because the controls are supposed to be constant in $[\tau_0, \tau_s[$, contact points $\tau_c \in]\tau_0, \tau_s[$ are touch points.

The line of argument given above can be analogously applied to the subintervals $[\tau_s, \tau_s + \Delta\tau[$ and $[\tau_s + \Delta\tau, \tau_f]$ as well as to phase constraints $C(x, u) \leq 0$. Another consequence is, that – again in general – not more than n contact points are possible in the subinterval $[\tau_0, \tau_s[$.

If τ_c coincides with τ_0, τ_s, $\tau_s + \Delta\tau$ or τ_f, the conditions

$$\begin{aligned} C(x(\tau_c)) &= 0\,, \\ \frac{\mathrm{d}}{\mathrm{d}t} C(x(t))\bigg|_{t=\tau_c} = C_x(x(\tau_c))^T f(x(\tau_c), u(\tau_c)) &= 0\,. \end{aligned}$$

are added to the interior point conditions in (14) at $t = \tau_c$.

If τ_c does not coincide with τ_0, τ_s, $\tau_s + \Delta\tau$ or τ_f, the classical solution procedure [14] is not applicable here: Because no control is left in the system described by (14), the constraint qualification [11] is violated.

Let now be $\tau_c \in]\tau_0, \tau_s[$. To overcome the problem mentioned above, τ_c is defined as an additional node that subdivides $]\tau_0, \tau_s[$ into two new intervals. Here the following interior point conditions hold

$$x(\tau_c^-) = x(\tau_c^+)\,, \tag{21}$$

$$w(\tau_c^-) = w(\tau_c^+)\,, \tag{22}$$

$$C(x(\tau_c)) = 0\,, \tag{23}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} C(x(t))\bigg|_{t=\tau_c} = C_x(x(\tau_c))^T f(x(\tau_c), u(\tau_c)) = 0\,. \tag{24}$$

Application of the generalized first order necessary conditions (first variation of $\tilde{I}$) analogously to Sect. 3.2 leads to a numerically stable formulation of the missing interior point conditions for the touch point τ_c.

3.4 Estimation of the control structure

The problem analyzed so far has the following control structure:

The interval $[\tau_0, \tau_f]$ is subdivided by

$$\tau_0 < \tau_1 < \tau_1 + \Delta\tau < \tau_2 < \tau_2 + \Delta\tau < \ldots < \tau_p < \tau_p + \Delta\tau < \tau_{p+1} \equiv \tau_f\,, \quad p \in \mathbb{N}\,. \tag{25}$$

The control functions are constant on every subinterval

$$u_k(t) = u_k^{\langle i\rangle} \in \mathbb{R} \;\; \forall\, t \in [\tau_i + \Delta\tau, \tau_{i+1}[\;\text{ and }\; u_k(t) = u_k^{\langle 0\rangle} \in \mathbb{R} \;\; \forall\, t \in [\tau_0, \tau_1[\,, \;\; i = 1,\ldots,p\,, \;\; k = 1,\ldots,n\,. \tag{26}$$

During the time period $[\tau_i, \tau_i + \Delta\tau[\,,\ i = 1,\ldots,p$, some controls remain constant and others switch from one value to another. The switching process is not instantaneous, but modeled by given profile functions S_{ik}.

Optimal values for $u_k^{\langle i\rangle} \in [u_k^-, u_k^+]$ and optimal switching times τ_i can be computed using the formalism set up in Sect. 3.2-3.3.

In many technical applications, the freedom of choice of the controls is further restricted. The piecewise constant controls $u_k^{\langle i\rangle}$ can no longer be optimized within the full range $[u_k^-, u_k^+]$, but have to be selected from a finite set of given control levels

$$u_k^{\langle i\rangle} \in \left\{u_{k,1},\ldots,u_{k,\kappa_k}\right\}, \quad k = 1,\ldots,n\,,\;\; \kappa_k \in \mathbb{N}\,,\;\; i = 0,\ldots,p, \tag{27}$$

with $u_{k,1} := u_k^-$, $u_{k,\kappa_k} := u_k^+$. The set of possible controls is non-convex. Because the positions of the nodes τ_i and the values of the state variables $x(\tau_i)$ are still subject to (nonlinear) optimization, the result is a constrained and nonlinear mixed-integer optimization problem. It can be embedded into the formalism of the calculus of variations by the methods developed in Sect. 3.2-3.3. The only modification is the inclusion of additional interior point conditions

$$u_k(\tau_i^+) = u_{k,j}\,, \quad k = 1,\ldots,n\,,\;\; j \in \{1,\ldots,\kappa_k\}\,,\;\; i = 0,\ldots,p\,. \tag{28}$$

While the vectors u and w respectively remain constant on a single interval, a large number of tiny intervals with different control structures may occur along the solution trajectory. As soon as the solution structure – i.e. only the sequence of the different control levels, but not the interval lengths $(\tau_{i+1} - \tau_i)$ – is known, the associated boundary value problem is completely defined and can be solved numerically [5].

To obtain the required information on the solution structure, the associated infinite-dimensional problem – e.g. (7,8) in case of the demonstrator problem – without the restrictions (26, 28) on the controls is solved first. The optimal solution $(x^\star(t), u^\star(t))$ is then used as a reference solution for the finite-dimensional problem.

The comparison between $(x^\star(t), u^\star(t))$ and the solution $(x^{\star\star}(t), u^{\star\star}(t))$ of the finite-dimensional problem is motivated by Bellman's recursion [1, 2], which is exemplified at the following special optimal control problem of the Lagrange type:

Find a piecewise differentiable state function $x \in \mathscr{C}_c^1\left([\tau_0, \tau_f], \mathbb{R}^{2n}\right)$ and a piecewise continuous control function $u \in \mathscr{C}_p^0\left([\tau_0, \tau_f], U \subseteq \mathbb{R}^p\right)$, which minimize

$$I(u) := \int_{\tau_0}^{\tau_f} L(x(t), u(t), t)\, \mathrm{d}t$$

subject to the conditions $\dot{x}(t) = f(x(t), u(t))\,,\ x(\tau_0) = x_0$. Let L, F be sufficiently often differentiable and let an optimal solution $(x^\star(t), u^\star(t))$ exist. Let $V(\hat{x}, \hat{t})$ be defined as the minimum of $I(u)$, if the initial values (x_0, τ_0) are replaced by $(\hat{x}, \hat{t})$. Let $h > 0$ be sufficiently small and define $x(s)$, $s \in [\hat{t}, \hat{t} + h]$, as the solution of the initial value problem $\dot{x}(s) = f(x(s), u(s))\,,\ x(\hat{t}) = \hat{x}$. Then one gets [1, 2]

$$V(\hat{x}, \hat{t}) = \min_{\substack{u(s) \in U \\ \hat{t} \le s \le \hat{t}+h}} \left[\int_{\hat{t}}^{\hat{t}+h} L(x(s), u(s))\, \mathrm{d}s + V(x(\hat{t}+h), \hat{t}+h)\right].$$

$\diamondsuit$

This motivates a strategy that keeps $x^{\star\star}(t)$ as close as possible to $x^\star(t)$ by choosing the control $u^{\star\star}(t)$ as close as possible to $u^\star(t)$ in such a way that $u^\star(t)$ and $u^{\star\star}(t)$ produce the same integral effect on the system in finite time intervals.

Application to the example problem with $x^\star(\hat{t}) = x^{\star\star}(\hat{t})$ yields for $t \in [\hat{t}, \hat{t} + h]$

$$x^{\star\star}(t) - x^\star(t) \;\; \doteq \;\; \int_{\hat{t}}^{t} \left(f_x(x^\star, u^\star)(x^{\star\star} - x^\star) + f_u(x^\star, u^\star)(u^{\star\star} - u^\star)\right) \mathrm{d}s\,,$$

$$\begin{aligned} &\int_{\hat{t}}^{\hat{t}+h} \left(L(x^{\star\star}(s),u^{\star\star}(s)) - L(x^{\star}(s),u^{\star}(s))\right) \mathrm{d}s \\ \doteq\ &\int_{\hat{t}}^{\hat{t}+h} \left(L_x(x^{\star},u^{\star})(x^{\star\star}-x^{\star}) + L_x(u^{\star},u^{\star})(u^{\star\star}-u^{\star})\right) \mathrm{d}s\,. \end{aligned}$$

Because mechanical systems react with a certain inertia to control inputs, one may assume

$$\|x^{\star}(t) - x^{\star\star}(t)\| \ll \|u^{\star}(t) - u^{\star\star}(t)\| \quad \forall t \in [\hat{t}, \hat{t}+h]\,.$$

With this assumption, the primary focus is on minimizing the quantity

$$\left| \int_{\hat{t}}^{\hat{t}+h} (u^{\star}(s) - u^{\star\star}(s))\,\mathrm{d}s \right|\,. \tag{29}$$

This idea is directly transferred into a numerical algorithm. It is outlined here for the case of one piecewise constant control $u(t)$ which can take only discrete values. The situation is sketched in Fig. 1; u_1 and u_2 denote two consecutive discrete levels. Starting with t_1 and t_2, for which $u^{\star}(t_1) = u^{\star\star}(t_1) = u_1$, $u^{\star}(t_2) = u^{\star\star}(t_2) = u_2$ and

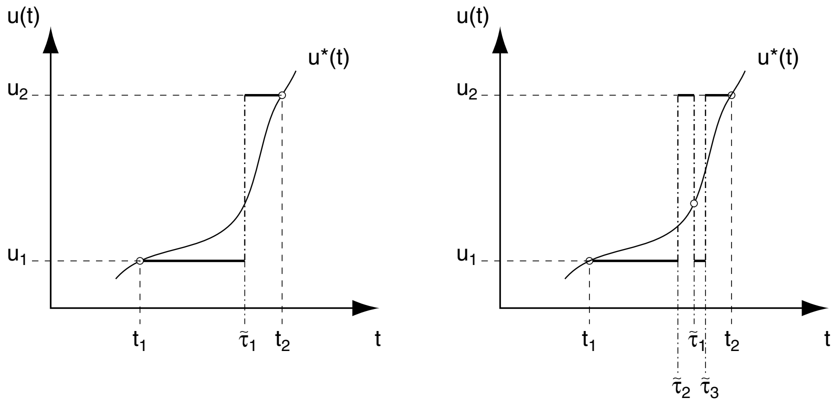

Figure 1: Successive estimation of the control structure.

$u^{\star}(t) \in]u_1, u_2[\ \ \forall t \in]t_1, t_2[$, an intermediate time $\tilde{\tau}_1$ is calculated (Fig. 1, left), such that

$$\int_{t_1}^{\tilde{\tau}_1} (u^{\star}(\xi) - u_1)\,\mathrm{d}\xi = \int_{t_2}^{\tilde{\tau}_1} (u_2 - u^{\star}(\xi))\,\mathrm{d}\xi\,. \tag{30}$$

Each newly created subinterval is further subdivided analogously (Fig. 1, right), e.g. $\tilde{\tau}_2$ is determined by

$$\int_{t_1}^{\tilde{\tau}_2} (u^{\star}(\xi) - u_1)\,\mathrm{d}\xi = \int_{\tilde{\tau}_1}^{\tilde{\tau}_2} (u_2 - u^{\star}(\xi))\,\mathrm{d}\xi\,. \tag{31}$$

The procedure resembles the adaptive Simpson method in numerical integration [16]. The $\tilde{\tau}_j$ are then rearranged in increasing size and form a subset of $\{\tau_i \,|\, i = 1, \ldots, \tilde{m}\}$.

Remarks. After each refinement step of the control structure, an optimal control problem is solved to adjust the τ_i with high precision before the next subdivison step. Therefore, small inaccuracies in the determination of the estimates for the τ_i by the algorithm sketched above play no role at all for the final accuracy. They only affect the convergence rate of the numerical solution of that intermediate control problem. The important lesson learned from Bellman's recursion is, that the optimal control has to be approximated by a repeated switching between the two nearest permissible control levels enclosing the optimal solution $x^{\star}(t)$ of the infinite-dimensional problem.

The permanent comparison between $x^{\star}(t)$ and the solution of the finite-dimensional problem for the current switching structure provides an excellent and reliable stopping criterion for further refinements.

3.5 Remarks concerning the numerical treatment

The differential equations for the adjoint variables are calculated recursively and simultaneously with the recursive calculation of the equations of motion by a Newton-Euler type algorithm. Even higher-order derivatives introduced by state constraints are generated recursively. For an efficient numerical treatment, structural information about the robotic system is used to a great extent. Nonlinear state and control constraints are treated without any simplification by transforming them into linear systems. These topics are discussed in detail in [7, 8, 9].

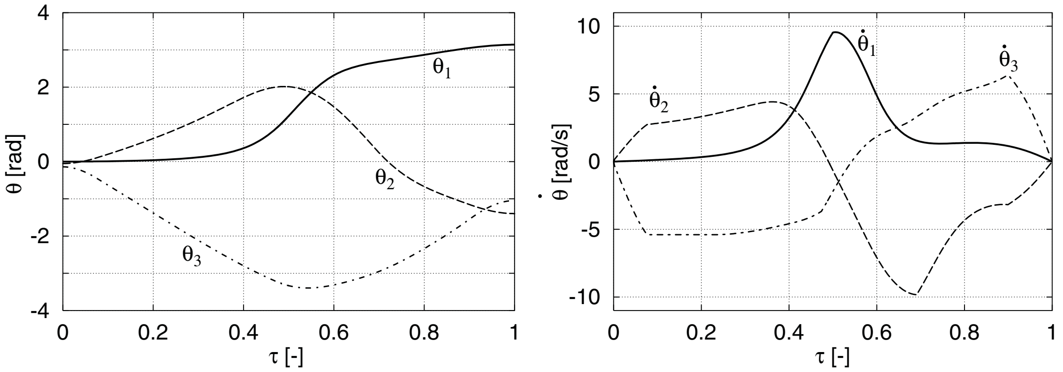

Figure 2: Infinite-dimensional reference problem: Link angles θ_i and link velocities $\dot{\theta}_i$ vs. time for the example problem.

The numerical treatment of the multi-point boundary value problems is by the advanced version JANUS [5] of the multiple shooting method [4]. It provides an improved stability of the solution process and an increased rate of convergence. One of the key features of *JANUS* is the efficient and accurate handling of a large number of interior points without increasing the overall size of the multiple shooting system. A detailed description is given in [5].

4 Numerical example

As an example, a time-optimal three-dimensional trajectory subject to one state constraint is calculated for a three-link manipulator ($n = p = 3$). The selection of the ratios of the maximum joint torques is non-standard.

	Unit	Variable	Link 1	Link 2	Link 3	Link 4
Mass	[kg]	m	1.00	1.00	1.00	1.00
Center of mass	[m]	P_x	0.50	0.50	0.00	0.00
		P_y	0.00	0.00	0.50	0.00
		P_z	0.50	0.00	0.00	0.00
Moments of inertia	[kg m^2/12]	I_{xx}	1.00	1.00	1.00	0.00
		I_{yy}	1.00	1.00	1.00	4.00
		I_{zz}	1.00	1.00	1.00	0.00
Denavit-Hartenberg parameters	[m]	a	0.00	0.50	1.00	0.00
	[rad]	α	π	$\frac{\pi}{2}$	0.00	$-\frac{\pi}{2}$
	[m]	d	-1.00	0.00	0.00	1.00
	[rad]	θ	θ_1	θ_2	$\theta_3 - \frac{\pi}{2}$	π

Table 1: Data for the example manipulator.

The manipulator geometry and the definition of the link systems are defined in detail in [9]. Important manipulator data and the Denavit-Hartenberg parameters are summarized in Table 1.

Motor torques are limited by

$$\begin{array}{rcccl} -10\,[\mathrm{Nm}] & \leq & T_1 & \leq & 10\,[\mathrm{Nm}]\,, \\ -20\,[\mathrm{Nm}] & \leq & T_2 & \leq & 20\,[\mathrm{Nm}]\,, \\ -20\,[\mathrm{Nm}] & \leq & T_3 & \leq & 20\,[\mathrm{Nm}]\,. \end{array}$$

The angular velocity of link 3 ist limited by

$$\dot{\theta}_3 \geq -5.4\,[\mathrm{rad/s}]\,.$$

In the infinite-dimensional case, this is a state constraint of order 1.

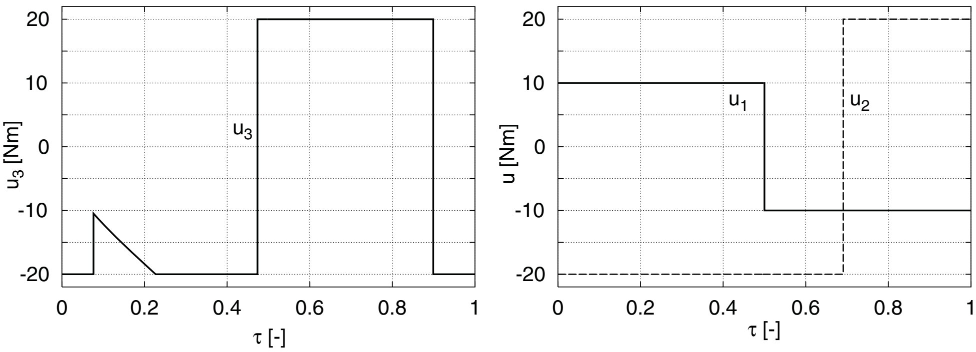

Figure 3: Infinite-dimensional reference problem: Motor torques u_i vs. time for the example problem.

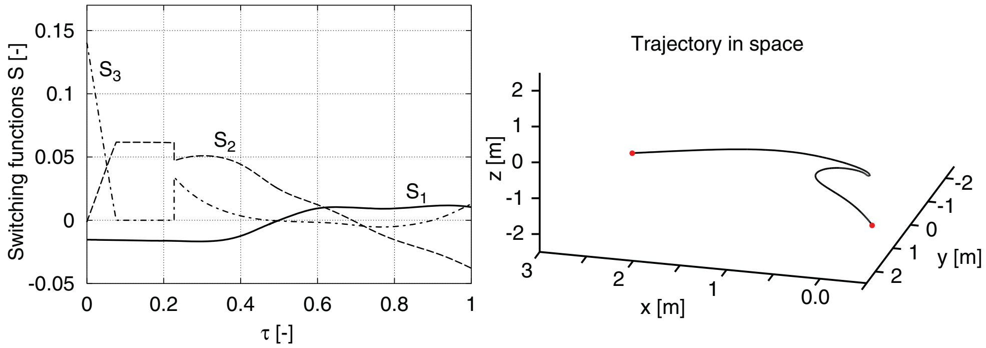

Figure 4: Infinite-dimensional reference problem: Switching functions S_i vs. time and trajectory in space.

For the fixed initial and final conditions

$$
\begin{aligned}
\theta_1(0) &= 0.0\,, & \theta_1(t_f) &= 3.1415927\,,\\
\theta_2(0) &= -0.052359877\,, & \theta_2(t_f) &= -1.3962633\,,\\
\theta_3(0) &= -0.139626340\,, & \theta_3(t_f) &= -1.0471975\,,\\
\dot\theta_1(0) &= 0.0\,, & \dot\theta_1(t_f) &= 0.0\,,\\
\dot\theta_2(0) &= 0.0\,, & \dot\theta_2(t_f) &= 0.0\,,\\
\dot\theta_3(0) &= 0.0\,, & \dot\theta_3(t_f) &= 0.0\,,
\end{aligned}
$$

the time-optimal point-to-point reference trajectory of the infinite-dimensional optimal control problem is calculated first. A final time of $t_f = 1.38$ [s] (more "exact" for comparison purposes: $t_f = 1.37975$) is achieved. The state constraint is fulfilled with a rel. precision of 10^{-9}.

Important properties of the solution like the time behaviour of the link angles, link velocities, motor torques, switching functions and the trajectory in space are depicted in Fig. 2-4.

In case of discrete control levels, the major part of the solution remains unchanged. Bang-bang control already dominates the time-optimal problem in the infinite-dimensional case. In the example problem, ten discrete control levels are permitted for each control. $\Delta\tau := 20$ [ms] and the profile functions are linear. For u_1 and u_2, only two levels each are active. Four levels are active for u_3. With ten subintervals for u_3 and two for u_1 and u_2 each, the final time of the optimal solution is by a factor of 1.00084 greater than for the infinite-dimensional reference solution. The results are shown in Fig. 5. Five touch points at the state constraint are observed. The order of the state constraint is not decisive in this case. Visually no difference can be detected between the trajectory in case of discrete controls and that of the infinite-dimensional reference solution.

5 Conclusions

A Maximum Principle based approach is used to calculate optimal trajectories with high accuracy for manipulators with discrete controls. Transitions between different control levels are not idealized by step functions, but described more realistically by continuous transition profiles. The original optimal control problem is embedded into a piecewise defined, continuous control problem of much higher dimension. By this, system dynamics and sensitivity properties on the full solution interval are coupled to the interior switching points. Significant drawbacks of the indirect approach have been resolved. Derivative information is generated in a structured and efficient way, the modeling of the optimal control problem and of the associated boundary value problem is automated to a great extent. Advanced numerical algorithms provide a significantly improved stability of the solution process. Applying embedding techniques to the discrete problem and permanently comparing it with the solution of the corresponding continuous (i.e. infinite-dimensional) problem proves to be a very powerful strategy for the perfor-

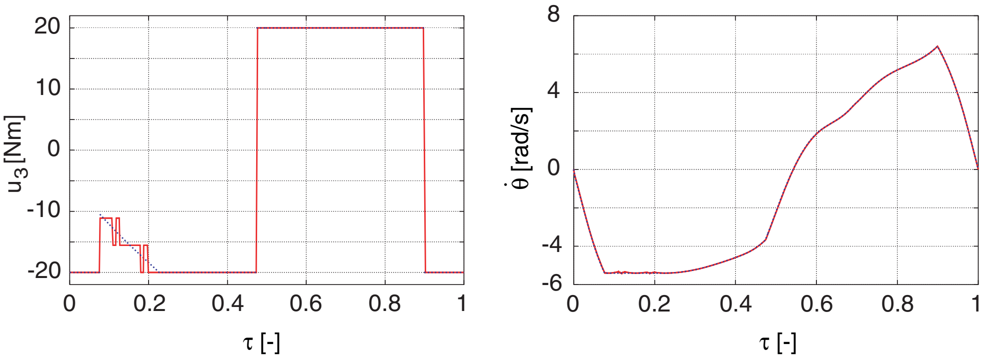

Figure 5: Finite-dimensional problem: State constrained solution for at maximum 10 discrete control levels (red); four control levels are active. Motor torque u_3 and link velocity $\dot{\theta}_3$ vs. time for the example problem. The solution of the infinite-dimensional case is given for comparison purposes (blue).

mance optimization of manipulators with discrete controls. The new approach is especially attractive, if discrete and continuous controls shall be treated in a common framework with high accuracy.

Optimal three-dimensional motion trajectories are calculated for a 3-DOF example system. With discrete controls properly calculated, a similar performance of a robotic manipulator can be achieved as with continuous controls. However, the solution structure is far more complicated than in the continuous case, especially if additional state constraints are active.

6 References

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