PORT-HAMILTONIAN FORMULATION OF PHYSICAL SYSTEMS CONTAINING MEMRISTORS

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Abstract. The port-Hamiltonian modeling framework is extended to a class of systems containing memristive elements. First, the concept of memristance is generalized to the same generic level as the port-Hamiltonian framework. Secondly, the underlying Dirac structure is augmented with a memristive port. Since both an integration and a differentiation is involved in this process, a memristor, like a resistor, appears to be causally neutral. This leaves two possible choices to configure the memristive port. The inclusion of memristive elements in the port-Hamiltonian framework turns out to be almost as straightforward as the inclusion of resistive elements. Although a memristor is a resistive element, it is also a dynamic element since the associated Ohmian laws are rather expressed in terms of differential equations. This means that the state space manifold, as naturally defined by the storage elements, is augmented by the states associated to the memristive elements. Hence the order of complexity is, in general, defined by the number of storage elements plus the number of memristors in the system. Apart from enlarging our repertoire of modeling building blocks, the inclusion of memristive elements in the existing port-Hamiltonian formalism possibly opens up new ideas for controller synthesis and design.

1 Introduction and Motivation

In the early seventies, Chua [3] postulated the existence of a new basic electrical circuit element, called the memristor, defined by a nonlinear relationship between charge and flux-linkage. The memristor, a contraction of memory and resistance that refers to a resistor with memory, completes the family of the well-known existing fundamental circuit elements: the resistor, inductor, and capacitor. Although a variety of physical devices, including thermistors, discharge tubes, Josephson junctions, and even ionic systems like the Hodgkin-Huxley model of a neuron, were shown to exhibit memristive effects [4, 5], a physical passive two-terminal memristive prototype could not be constructed until very recently scientists of Hewlett-Packard Laboratories announced its realization in Nature [10]. Strukov et al. show that memristance naturally arises in nanoscale systems when electronic and atomic transport are coupled under an external bias voltage. On the other hand, as pointed out in [9], a tapered dashpot is a mechanical resistor whose resistance depends on the displacement of its terminals. Consequently, a description in terms of its associated force and velocity generally yields some complicated, possibly hysteretic, constitutive relationship. These difficulties are circumvented by modeling the tapered dashpot as a mechanical memristive element using the relationship between its displacement and momentum (the mechanical analogies of charge and flux linkages) instead.

One of the main reasons why the memristor concept has not yet played a major role in modeling problems can most likely be explained from the fact that so far the majority of practical devices are reasonably well modeled by some (though often artificial) combination of standard modeling building blocks, like resistive, inductive, and capacitive elements, and their nonlinear and multiport versions. However, as nanoscale electronic devices become more and more important and complex [4], it might be beneficial, and on the longer term even necessary, to enlarge our repertoire of modeling building blocks that establishes a closer connection between the mathematics and the observed physics.

In this paper, we study the inclusion of memristive elements and their properties in the port-Hamiltonian modeling framework. The port-Hamiltonian formalism naturally arises from network modeling of physical systems in a variety of domains (e.g., mechanical, electrical, electromechanical, hydrodynamical, and thermodynamical). Exposing the relation between the energy storage, dissipation, and interconnection structure, this framework underscores the physics of the system. The connection with network (bond-graph) modeling is further formalized with the notion of a so-called Dirac structure on the space of flows and efforts. One of the strong aspects of the port-Hamiltonian formalism is that a power-preserving interconnection between port-Hamiltonian systems results in another port-Hamiltonian system with composite energy, dissipation, and interconnection structure. Based on this principle, complex, multidomain systems can be modeled by interconnecting port-Hamiltonian descriptions of its subsystems. Moreover, several control design methodologies are available that can be directly applied to such port-Hamiltonian descriptions of complex nonlinear systems. It is precisely in this context that a memristive port-Hamiltonian description can be of added value.

The remainder of the paper is organized as follows. In Section 2, we briefly recall the basic properties of port-Hamiltonian systems defined with respect to a Dirac structure. Section 3 gives the generalization of the concept of memristance to the same generic level as the port-Hamiltonian framework. The extension of the input-state-output port-Hamiltonian formulation with a generalized memristive port and some of its basic properties are highlighted in Section 4. Section 5 illustrates some aspects of the theory by using three simple examples. The paper is concluded with a slight extension of the framework to allow for mutual dependencies between some of the memristive elements and dependencies between some of the memristive elements and the environment, and some final remarks.

Notation. All vectors, including the gradient of a function, defined in the paper are column vectors.

2 The Port-Hamiltonian Formalism

The basic ingredient of any port-Hamiltonian system is the power-conserving interconnection structure, mathematically formalized as a Dirac structure, linking the various power ports of the system, see Figure 1. Power ports (henceforth simply called ports) carry two sets of conjugate variables: a vector of flow variables $f \in \mathscr{F}$ and a vector of effort variables $e \in \mathscr{E}$, with product $e^T f$ denoting the power occurring at the port. The Dirac structure captures the basic interconnection laws (like Kirchhoff's laws) together with ideal power-conserving elements like transformers, gyrators, and ideal constraints, and generalizes Tellegen's theorem and d'Alembert's principle.

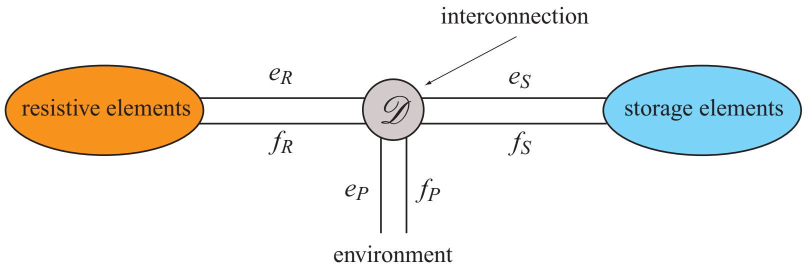

Figure 1: Many physical systems can be characterized by interconnections between energy storage elements, resistive elements, and the environment. The key concept in the formulation of port-based network models of physical systems as port-Hamiltonian systems is the geometric notion of a Dirac structure $\mathscr{D}$.

In contrast to the common modeling approaches, where two dual types of energy storage (like kinetic and potential energy in mechanical systems, or electric and magnetic energy in electrical networks) are distinguished, the definition of the energy storage port in the port-Hamiltonian framework assumes just one type of storage. This approach is adopted from the so-called Generalized Bond Graph (GBG) framework introduced in [1]. In this framework the usual physical domains are split into two subdomains that are explicitly connected by a so-called symplectic gyrator. Consequently, we do not speak of mechanical or electrical domains, but of kinetic and potential, or electric and magnetic domains, etc., see Table 1 for a complete overview.[1]

2.1 Ports, Dirac Structures, and Passivity

In order to define a Dirac structure, the spaces of flows and efforts are naturally partitioned as $\mathscr{F} := \mathscr{F}_S \times \mathscr{F}_R \times \mathscr{F}_P$ and $\mathscr{E} := \mathscr{E}_S \times \mathscr{E}_R \times \mathscr{E}_P$, each corresponding to the following set of ports:

- The energy storage port, with port variables $(f_S, e_S) \in \mathscr{F}_S \times \mathscr{E}_S$, is interconnected with the energy storage of the system, which in turn is characterized by an n_S-dimensional space $\mathscr{X}$ of state variables, locally represented by $x \in \mathscr{X}$, together with a Hamiltonian function $H : \mathscr{X} \to \mathbb{R}$ denoting the total stored energy. The corresponding flow variables are given by the rate of change of the state variables. This is accomplished by setting

$$\begin{aligned} f_S &= -\dot{x} \\ e_S &= \frac{\partial H}{\partial x}(x). \end{aligned} \tag{1}$$

Hence, the power at the energy storage port can be written as

$$\dot{H}(x) = \left(\frac{\partial H}{\partial x}(x)\right)^T \dot{x} = -e_S^T f_S. \tag{2}$$

[1] An additional advantage of the GBG framework is that the concept of mechanical force has no unique meaning as it may play the role of a flow in the kinetic domain or an effort in the potential domain, thus leaving the discussion about the force-voltage versus force-current analogy a non-issue.

Table 1: Domains and variables used in the port-Hamiltonian framework.

physical domain	flow $f \in \mathscr{F}$	effort $e \in \mathscr{F}$	state variable $x = \int f \mathrm{d}t$
electric	current	voltage	charge
magnetic	voltage	current	flux linkage
potential translation	velocity	force	displacement
kinetic translation	force	velocity	momentum
potential rotation	angular velocity	torque	angular displacement
kinetic rotation	torque	angular velocity	angular momentum
potential hydraulic	volume flow	pressure	volume
kinetic hydraulic	pressure	volume flow	flow tube momentum
chemical	molar flow	chemical potential	number of moles
thermal	entropy flow	temperature	entropy

- The resistive port corresponds to internal energy dissipation (e.g., friction, electrical resistance, etc.), and its port variables $(f_R, e_R) \in \mathscr{F}_R \times \mathscr{E}_R$ are terminated by a static resistive relation of the form

$$f_R = -F_R(e_R), \tag{3}$$

  with $F_R : \mathscr{E}_R \to \mathscr{F}_R$. In many cases, F_R can be derived from a so-called *content* function $D : \mathscr{E}_R \to \mathbb{R}$ in the sense that $F_R(e_R) = \frac{\partial D}{\partial e_R}(e_R)$. For linear resistive elements, (3) reduces to $f_R = -R_e e_R$, with $R_e = R_e^T$ some constant resistance matrix. Note that for passive resistors $e_R^T F_R(e_R) \geq 0$, or equivalently, $e_R^T f_R \leq 0$.
- Finally, the remaining port, with port variables $(f_P, e_P) \in \mathscr{F}_P \times \mathscr{E}_P$, denotes the interaction port of the system, modeling its interaction with other system components or the environment.

The Dirac structure $\mathscr{D}$ is a linear relation between all the port variables that satisfy the power-conservation property

$$e_S^T f_S + e_R^T f_R + e_P^T f_P = 0, \tag{4}$$

and has maximal dimension with respect to this property.[2] It follows that any port-Hamiltonian system with passive resistive elements satisfies the power-balance inequality

$$\dot{H}(x) = -e_S^T f_S = e_R^T f_R + e_P^T f_P \leq e_P^T f_P \tag{5}$$

since $e_R^T f_R \leq 0$. Integrating the latter from initial time t_0 to t yields the energy-balance inequality

$$H[x(t)] - H[x(t_0)] = \int_{t_0}^{t} e_R^T(t') f_R(t') \mathrm{d}t' + \int_{t_0}^{t} e_P^T(t') f_P(t') \mathrm{d}t' \leq \int_{t_0}^{t} e_P^T(t') f_P(t') \mathrm{d}t'. \tag{6}$$

If the Hamiltonian function $H(x)$ is bounded from below, then port-Hamiltonian systems are passive with respect to the supply rate $e_P^T f_P$ and storage function the Hamiltonian function. Note that, recalling Lyapunov stability theory, together with the sufficient conditions for the stability of an equilibrium point, it can shown that the Hamiltonian is often a bona-fide candidate Lyapunov function [11].

2.2 Input-State-Output Representation

An important special case of port-Hamiltonian systems is the class of input-state-output port-Hamiltonian systems, where there are no algebraic constraints on the state variables, and the flow and effort variables at all the other ports have been split into conjugated input-output pairs. The corresponding Dirac structure, in kernel representation, is defined by

$$\begin{aligned} \mathscr{D} = \Big\{ (f_S, e_S, f_R, e_R, f_P, e_P) \in \mathscr{F} \times \mathscr{E} \,\big|\, & f_S + J e_S + G_R f_R + G_P f_P = 0, \\ & -G_R^T e_S + e_R = 0, -G_P^T e_S + e_P = 0 \Big\}, \end{aligned} \tag{7}$$

where $J = -J^T$, G_R, and G_P are matrices of appropriate dimensions depending on the interconnection, resistive, and input-output structure of the system, respectively. Furthermore, assuming that the resistive elements (3) are linear, the constitutive relationship (3) simplifies to

$$f_R = -R_e e_R, \tag{8}$$

[2] Note that (4) is a direct generalization of Tellegen's theorem.

with $R_e = R_e^T$ some constant resistance matrix. Then, around $x \in \mathscr{X}$, by utilizing (1) and (3), the dynamics on $\mathscr{D}$ take the form

$$-\dot{x} + J\frac{\partial H}{\partial x}(x) - G_R R_e e_R + G_P f_P = 0,$$
$$-G_R^T \frac{\partial H}{\partial x}(x) + e_R = 0,$$
$$-G_P^T \frac{\partial H}{\partial x}(x) + e_P = 0,$$

which, after substitution of the second equation into the first and a slight rearrangement, yields the well-known input-state-output port-Hamiltonian representation

$$\begin{aligned} \dot{x} &= \big(J - R\big)\frac{\partial H}{\partial x}(x) + G_P f_P \\ e_P &= G_P^T \frac{\partial H}{\partial x}(x), \end{aligned} \tag{9}$$

with resistive structure matrix $R := G_R R_e G_R^T$. Consequently, the power-balance inequality (5) can be written as

$$\dot{H}(x) = -\left(\frac{\partial H}{\partial x}(x)\right)^T R\frac{\partial H}{\partial x}(x) + e_P^T f_P \leq e_P^T f_P, \tag{10}$$

under the condition that $R \succeq 0$. Note that in this framework, the flow and effort related to the environment are naturally defined as the input and output of the system, respectively.

For many systems, especially those with 3D mechanical components, the Dirac structure will in general be modulated by the state variables x. In such a case, the structure matrices J, G_R, and G_P are replaced by their modulated versions $J(x)$, $G_R(x)$, and $G_P(x)$, respectively. More details on the geometric properties of Dirac structures and port-Hamiltonian systems can be found in [2, 6, 11].

3 Properties of the Memristor

Before generalizing the concept of memristance to fit the definitions of the port-Hamiltonian framework discussed in the previous section, we will first briefly recall the basic properties of the electrical memristor.

3.1 Chua's Memristor

Since electronics was developed, engineers designed circuits using combinations of three basic two-terminal elements: resistors, inductors, and capacitors. From a mathematical perspective, the behavior of each of these elements, whether linear or nonlinear, is described by relationships between two of the four basic electrical variables: voltage, current, charge, and flux linkage. A resistor is described by the relationship of current and voltage; a capacitor by that of voltage and charge, and an inductor by that of current and flux linkage. But what about the relationship between charge and flux linkage? As argued by Chua in the early seventies [3], a fourth element should be added to complete the symmetry. He coined this 'missing element' the *memristor*, referring to a resistor with memory. The memory aspect stems from the fact that a memristor 'remembers' the amount of current that has passed through it together with the total applied voltage. More specifically, if q denotes the charge and ϕ denotes the flux linkage, then a two-terminal *charge-controlled* memristor is defined by the constitutive relationship

$$\phi = \hat{\phi}(q).$$

Since flux linkage is the time integral of voltage u (like in Faraday's law), and charge is the time integral of current i, or equivalently, $u = \dot{\phi}$ and $i = \dot{q}$, we obtain

$$u = M_i(q)i, \tag{11}$$

where $M_i(q) := \mathrm{d}\hat{\phi}(q)/\mathrm{d}q$ is called the incremental *memristance*.

Similarly, a two-terminal *flux-controlled* memristor (memductor) is defined by

$$q = \hat{q}(\phi),$$

Differentiation yields the dual of (11),

$$i = M_u(\phi)u, \tag{12}$$

where $M_u(\phi) := \mathrm{d}\hat{q}(\phi)/\mathrm{d}\phi$ is called the incremental *memductance*.

Observe that (11) and (12) are just charge- and flux-modulated versions of Ohm's law, respectively. It is important to realize that for the special cases that the constitutive relations are linear, that is, when the incremental memristance M_i or the incremental memductance M_u is constant, a memristor or memductor becomes an ordinary resistor or conductor. Hence, memristors and memductors are only relevant in nonlinear circuits, which may account in part for their neglect in linear network and systems theory.

Before the effect of memristive elements can be studied in the port-Hamiltonian framework, we first need to bring the concept to the same generic level. This is accomplished by generalizing the constitutive relationships (11) and (12) to the level of flows and efforts.

3.2 The Generalized Memristor

In view of the classifications and analogies of Table 1, the generalization of either the charge-controlled memristor (11) or the flux-controlled memristor (12) is easily deduced as follows. Let $x_f \in \mathscr{X}_f$ denote the integrated flow, and let $x_e \in \mathscr{X}_e$ denote the integrated effort, or equivalently, $\dot{x}_f = f$, and $\dot{x}_e = e$, respectively, then the relationship

$$x_e = \hat{x}_e(x_f)$$

constitutes an x_f-controlled memristor, i.e.,

$$e = M_f(x_f)f, \tag{13}$$

with generalized memristance $M_f(x_f) := \mathrm{d}\hat{x}_e(x_f)/\mathrm{d}x_f$.

On the other hand, by interchanging the roles of the (integrated) flow and effort, we might as well consider the corresponding dual form

$$x_f = \hat{x}_f(x_e)$$

yielding an x_e-controlled memristor, i.e.,

$$f = M_e(x_e)e, \tag{14}$$

with generalized memristance $M_e(x_e) := \mathrm{d}\hat{x}_f(x_e)/\mathrm{d}x_e$.

In a similar fashion as the storage or resistive elements, the constitutive relationship of a memristive element will in many cases be derivable from a so-called memristive *action* function $A_f : \mathscr{X}_f \to \mathbb{R}$ (resp., $A_e : \mathscr{X}_e \to \mathbb{R}$) in the sense that

$$x_e = \frac{\partial A_f}{\partial x_f}(x_f) \quad \left(\text{resp., } x_f = \frac{\partial A_e}{\partial x_e}(x_e)\right). \tag{15}$$

More details on the action and some of its applications in a circuit-theoretic context can be found in [3, 8].

It is important to remark that the definitions (13) and (14) are in some sense arbitrary. This can be explained as follows. For energy storage elements the distinction between flow and effort as the equilibrium establishing (rate of change of state) and the equilibrium determining variable, respectively, is clear since a storage element is defined by a constitutive relationship between effort and integrated flow (state), or in a thermodynamic parlance, between an intensive state and an extensive state, i.e., $e = \hat{e}(x)$ or $x = \hat{x}(e)$, with $\dot{x} = f$. In terms of causality, the constitutive relationship $e = \hat{e}(x)$ yields a so-called integral causal form in which the flow can be considered as input and the effort as output, see Figure 2(a). This is the form generally considered in the port-Hamiltonian framework. The dual or co-energy form, $x = \hat{x}(e)$, yields a differential causal form, considering effort as input and flow as output. Clearly, since both an integration and a differentiation is involved in 'lifting' the memristor to the space of flows and efforts, the memristor, like the resistor, is causally neutral, i.e., there is no fixed or preferred causality, so that it can accept either a flow or an effort as input variable, see Figure 2(b). Furthermore, a memristor does not store integrated flow or integrated effort, it just book keeps the amount of integrated flow or integrated effort that passed its port. Hence it does not distinguish between the various subdomains outlined in Table 1. Moreover, the state variables associated to (13) and (14) are not defined state variables associated to any energy storage element in the system, but the state of the memristive element itself.

For ease of reference, we will refer to (13) as a flow-causal memristor and to (14) as an effort-causal memristor.

4 Port-Hamiltonian Systems with Memristive Dissipation

We are now ready to extend the port-Hamiltonian formalism, as introduced in Section 2, by adding a memristive port, with port variables $(f_M, e_M) \in \mathscr{F}_M \times \mathscr{E}_M$, to the Dirac structure. For simplicity, we first assume that the resistive port is vacuous, i.e., the system does not contain any resistive elements.

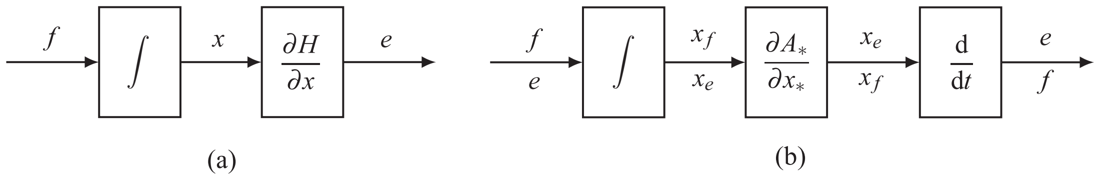

Figure 2: (a) Preferred causality of a storage element; (b) Causally natural character of a memristive element.

4.1 Input-State-Output Representation

Consider instead of (7) the Dirac structure

$$\begin{aligned}\mathscr{D} = \Big\{ (f_S, e_S, f_M, e_M, f_P, e_P) \in \mathscr{F} \times \mathscr{E} \,\big|\, f_S + Je_S + G_M f_M + G_P f_P = 0, \\ -G_M^T e_S + e_M = 0,\ -G_P^T e_S + e_P = 0 \Big\},\end{aligned} \tag{16}$$

where J and G_P are as before, and G_M is a matrix of appropriate dimensions depending on the memristive structure. Next, assuming that the memristive port variables are terminated by a flow-causal memristive constitutive relationship (cf. (13))

$$\begin{aligned} f_M &= \dot{x}_f, \\ e_M &= -M_f(x_f) f_M, \end{aligned} \tag{17}$$

with $M_f(x_f) = M_f^T(x_f)$ some memristance matrix of appropriate dimensions, the dynamics on $\mathscr{D}$ take the form

$$\begin{aligned} -\dot{x} + J\frac{\partial H}{\partial x}(x) + G_M \dot{x}_f + G_P f_P &= 0, \\ -G_M^T \frac{\partial H}{\partial x}(x) - M_f(x_f)\dot{x}_f &= 0, \\ -G_P^T \frac{\partial H}{\partial x}(x) + e_P &= 0. \end{aligned} \tag{18}$$

In a similar fashion as before, but with the additional condition that $\det\big(M_f(x_f)\big) \neq 0$, for all $x_f \in \mathscr{X}_f$, the latter set of equations can be rearranged into an input-state-output port-Hamiltonian system given by

$$\begin{aligned} \dot{x} &= \big(J - M(x_f)\big)\frac{\partial H}{\partial x}(x) + G_P f_P \\ e_P &= G_P^T \frac{\partial H}{\partial x}(x), \end{aligned} \tag{19}$$

with memristive structure matrix $M(x_f) := G_M M_f^{-1}(x_f) G_M^T$. Observe, however, that the formulation is incomplete without also considering the second equation of (18), i.e., the memristive state variable x_f will (in general) be determined by the solution of

$$\dot{x}_f = -M_f^{-1}(x_f) G_M^T \frac{\partial H}{\partial x}(x). \tag{20}$$

The invertibility condition on the memristance matrix can be avoided by starting from an effort-causal representation (cf. (14)) instead, i.e.,

$$\begin{aligned} f_M &= -M_e(x_e) e_M, \\ e_M &= \dot{x}_e, \end{aligned} \tag{21}$$

with $M_e(x_e) = M_e^T(x_e)$. Hence the dynamics on $\mathscr{D}$ now take the form

$$\begin{aligned} -\dot{x} + J\frac{\partial H}{\partial x}(x) - G_M M_e(x_e)\dot{x}_e + G_P f_P &= 0, \\ -G_M^T \frac{\partial H}{\partial x}(x) + \dot{x}_e &= 0, \\ -G_P^T \frac{\partial H}{\partial x}(x) + e_P &= 0, \end{aligned} \tag{22}$$

yielding again a port-Hamiltonian system of the form (19), but now with $M(x_e) := G_M M_e(x_e) G_M^T$, and

$$\dot{x}_e = G_M^T \frac{\partial H}{\partial x}(x). \tag{23}$$

4.2 Passivity and the Power-Balance Inequality

A memristive port described by (21) is passive if and only if its generalized memristance $M_e(x_e)$ is nonnegative, i.e., $M_e(x_e) \succeq 0$, for all $x_e \in \mathscr{X}_e$. Indeed, the instantaneous power dissipated by (21) is given by

$$P_M = e_M^T f_M = -e_M^T M_e(x_e) e_M \leq 0, \tag{24}$$

where we recall that the sign convention adopted here is that power supplied to the system carries a positive sign and power extracted from the system carries a negative sign.

The associated power-balance inequality takes the form

$$\dot{H}(x) = -\left(\frac{\partial H}{\partial x}(x)\right)^T M(x_e) \frac{\partial H}{\partial x}(x) + e_P^T f_P \leq e_P^T f_P, \tag{25}$$

where $M(x_e) := G_M M_e(x_e) G_M^T \succeq 0$ since $M_e(x_e) \succeq 0$, for all $x_e \in \mathscr{X}_e$. Hence if the Hamiltonian function $H(x)$ is bounded from below, then the system is passive with respect to the supply rate $e_P^T f_P$ and storage function the Hamiltonian function. Note that the additional dynamic equation (23) does not have any influence on the power-balance inequality other than that it provides a solution for x_e.

A similar discussion holds for a memristive port described by (17), where the additional invertibility condition implies that $M_f(x_f) \succ 0$, for all $x_f \in \mathscr{X}_f$.

4.3 Degenerate Case: Linear Memristance

We observe that, dynamically, the memristive port appears as either an integrated flow or an integrated effort modulated resistive port. In the linear case, i.e., when M_f in (17), or M_e in (21), is constant, the memristive port reduces to a purely resistive port. This property is consistent with the original definitions of the memristor outlined in Subsection 3.1.

4.4 Order of Complexity

The addition of the memristive port yields that the total state space is in general extended to either $\mathscr{X} \times \mathscr{X}_f$ or $\mathscr{X} \times \mathscr{X}_e$. Consequently, in addition to the initial values of the state variables associated to the storage elements, also the initial values of the memristors should be specified in order to find a complete solution of the port-Hamiltonian systems presented above. This means, in general, that the order of complexity [7] of a port-Hamiltonian system with memristive dissipation is determined by

$$n = n_S + n_M,$$

where n_S denotes the number of energy storage elements and n_M the number of memristive elements. As shown in the next section, depending on the system configuration the order of complexity can sometimes be reduced.

5 Examples

5.1 Josephson Junction Circuit Model

The classical circuit model for a Josephson junction consists of a parallel connection of a linear resistor r, a linear capacitor C, and a flux-controlled nonlinear inductor described by the constitutive relationship $i_L = I_o \sin(k\phi_L)$, where I_o is a device parameter and $k = 4\pi\varepsilon/\bar{h}$, with ε and $\bar{h}$ denoting the electron charge and Plank's constant, respectively. As discussed in [4], a more rigorous quantum mechanical analysis of the junction dynamics reveals the presence of an additional small current component that can be approximated by $i = g\cos(k_o\phi)u$, for some constants g, k_o. Obviously, the latter can be associated with a flux-controlled memristor (memductor) of the form

$$q = \frac{g}{k_o}\sin(k_o\phi),$$

with $\dot{q} = i$ and $\dot{\phi} = u$. Figure 3 shows the more realistic circuit model for a Josephson junction consisting of a parallel connection of each of the four basic circuit elements.

From a port-Hamiltonian perspective the circuit consists of four ports: an energy storage port defined by the total energy stored in the capacitor and the inductor, a memristive port, a resistive port, and an external port. The total stored energy is given by

$$H(q_C, \phi_L) = \frac{q_C^2}{2C} - \frac{I_o}{k}\cos(k\phi_L),$$

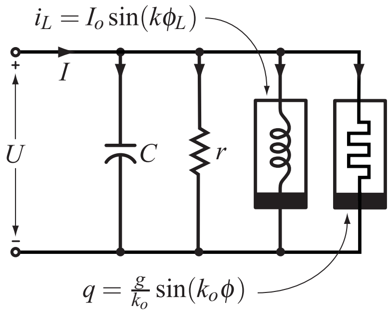

Figure 3: More realistic model of a Josephson junction [4].

which, according to Table 1, defines an energy storage port

$$
\begin{aligned}
-\dot{\phi}_L &= f_{S_1}\\
e_{S_1} &= \frac{\partial H}{\partial \phi_L}\\
-\dot{q}_C &= f_{S_2}\\
e_{S_2} &= \frac{\partial H}{\partial q_C}.
\end{aligned}
$$

Using the effort-causal form (21), the memristive port is defined by

$$
\begin{aligned}
f_M &= -g\cos(k_o\phi_e)e_M,\\
e_M &= \dot{\phi}_e,
\end{aligned}
$$

and according to Kirchhoff's laws we obtain the following set of structure matrices

$$
J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad G_M = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad G_P = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
$$

Although the presence of both resistive and memristive elements is not discussed explicitly, the system is easily extended by introducing a resistive port of the form $f_R = -e_R/r$ and setting $G_R = (0\ \ 1)^T$. On the other hand, since the resistor is linear it can also be considered as a degenerate memristor (see Subsection 4.3). However, in both cases the following input-state-output port-Hamiltonian system is obtained

$$
\begin{pmatrix} \dot{\phi}_L \\ \dot{q}_C \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -g\cos(k_o\phi_e) - \frac{1}{r} \end{pmatrix} \begin{pmatrix} \frac{\partial H}{\partial \phi_L} \\ \frac{\partial H}{\partial q_C} \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} I,
$$

together with

$$
\begin{aligned}
U &= \frac{\partial H}{\partial q_C},\\
\dot{\phi}_e &= \frac{\partial H}{\partial q_C}.
\end{aligned}
$$

Interestingly, the system is passive under the condition that $rg\cos(k_o\phi_e) \geq -1$, for all admissible ϕ_e.

5.2 Mechanical System

Consider the mechanical system depicted in Figure 4. The system consists of two carts with masses m_1 and m_2, interconnected by a linear spring with elastance k, and a tapered dashpot d. Since the storage elements are linear, we have $v_1 = p_1/m_1$, $v_2 = p_2/m_2$, and $F_k = kx_k$, where v_1, v_2, p_1, and p_2 are, respectively, the velocities and momenta of the two masses, and F_k and x_k are, respectively, the force and displacement of the spring. The total stored energy is given by

$$
H(p_1, p_2, x_k) = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + \frac{kx_k^2}{2}.
$$

According to Table 1, the energy storage port assumes the form

$$\begin{aligned} -\dot{p}_1 &= f_{S_1} \\ e_{S_1} &= \frac{\partial H}{\partial p_1} \\ -\dot{p}_2 &= f_{S_2} \\ e_{S_2} &= \frac{\partial H}{\partial p_2} \\ -\dot{x}_k &= f_{S_3} \\ e_{S_3} &= \frac{\partial H}{\partial x_k}. \end{aligned} \tag{26}$$

As argued in [9], a tapered dashpot can in principle not be treated as an ordinary damper since the incremental damping coefficient, i.e., the mechanical resistance, depends on the piston displacement. Hence a description in terms of its associated force F_d and velocity v_d generally yields some complicated (possibly hysteretic) constitutive relationship. These difficulties are circumvented by modeling the tapered dashpot as a memristive element. Indeed, suppose that the constitutive relationship is given by a monotonically increasing function $p_d = \hat{p}_d(x_d)$, where p_d and x_d denote the memristor's momentum and displacement, respectively, then $F_d = M_v(x_d)v_d$, with mechanical memristance $M_v(x_d) := \mathrm{d}\hat{p}_d(x_d)/\mathrm{d}x_d$, with $\dot{p}_d = F_d$ and $\dot{x}_d = v_d$. Hence the memristive port is defined (in effort-causal form) by

$$\begin{aligned} f_M &= -M_v(x_d)e_M, \\ e_M &= \dot{x}_d. \end{aligned} \tag{27}$$

Since there are no inputs and outputs, the interaction port is vacuous and $G_P = 0$. Furthermore, the interconnective relationships dictate the remaining structure matrices

$$J = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}, \quad G_M = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

Hence, the dynamics of the system are described by the following port-Hamiltonian equations

$$\begin{pmatrix} \dot{p}_1 \\ \dot{p}_2 \\ \dot{x}_k \end{pmatrix} = \underbrace{\begin{pmatrix} -M_v(x_d) & M_v(x_d) & -1 \\ M_v(x_d) & -M_v(x_d) & 1 \\ 1 & -1 & 0 \end{pmatrix}}_{J-M(x_d)} \begin{pmatrix} \frac{\partial H}{\partial p_1} \\ \frac{\partial H}{\partial p_2} \\ \frac{\partial H}{\partial x_k} \end{pmatrix}, \tag{28}$$

together with

$$\dot{x}_d = \begin{pmatrix} 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial H}{\partial p_1} \\ \frac{\partial H}{\partial p_2} \\ \frac{\partial H}{\partial x_k} \end{pmatrix}. \tag{29}$$

Differentiating the Hamiltonian $H(x)$, where $x = (p_1, p_2, x_k)^T$, along the trajectories of the system yields the power-balance of the system, i.e.,

$$\dot{H}(x) = -\left(\frac{\partial H}{\partial x}\right)^T M(x_d)\frac{\partial H}{\partial x} \le 0,$$

where the inequality stems from the fact that $M(x_d) \succeq 0$ by assumption. This implies that the mechanical system is passive—as should be expected.

Concerning the solvability of the port-Hamiltonian equations (28)–(29), at first glance the order of complexity for this system appears to be equal to four. However, since the relative velocity of the dashpot equals the difference between the velocities of the masses, its relative displacement is given by the integral

$$x_d(t) = x_d(t_0) + \int_{t_0}^{t} \left[\frac{\partial H}{\partial p_1}(p_1(t')) - \frac{\partial H}{\partial p_2}(p_2(t'))\right] \mathrm{d}t' = x_d(t_0) + \int_{t_0}^{t} \left[v_1(t') - v_2(t')\right] \mathrm{d}t',$$

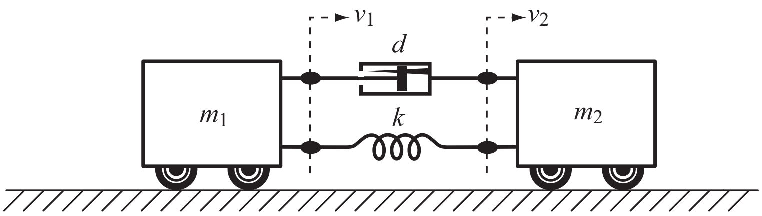

Figure 4: A mechanical mass-spring system with a tapered dashpot. Note that the shape of the pin may be machined to produce any desired memristance curve.

which implies that $x_d(t_0) = x_1(t_0) - x_2(t_0)$. This means that the initial value of the dashpot is determined by the initial distance between the two masses, which is a hard physical constraint.[3] The same discussion holds for the spring so that the actual order of complexity for this system equals two. Moreover, we have that $x_d = x_k$, which means that (29) can be eliminated by replacing x_d with x_k in (28).

It should be pointed out, however, that for this particular system it is a coincidence that it is possible to represent the tapered dashpot as a modulated resistor since its displacement coincides with the displacement of the spring, which, in turn, is proportional to the force in the spring. In general, the states of the memristive elements in a system are independent from the states of the energy storage elements, like in the Josephson junction circuit model. Another example of a system in which memristance plays a crucial role is the electrolytic tank system discussed in [9]. An example for which the minimal number of state equations is less than the order of complexity is briefly discussed next.

5.3 Electrical Network

Consider a flux-controlled memductor, with a constitutive relationship $q_M = \hat{q}_M(\phi_M)$, connected in parallel with a linear capacitor described by $u_C = q_C/C$. Following the ideas exposed in Section 4, we obtain the following port-Hamiltonian description

$$\begin{aligned} \dot{q}_C &= -M_u(\phi_M)\frac{\partial H}{\partial q_C}, \\ \dot{\phi}_M &= \frac{\partial H}{\partial q_C}, \end{aligned}$$

with $H(q_C) = q_C^2/(2C)$ and $M_u(\phi_M) := \mathrm{d}\hat{q}_M(\phi_M)/\mathrm{d}\phi_M$. Clearly, the system has two independent initial conditions $\phi_M(t_0)$ and $q_C(t_0)$. However, since $q_C(t) - q_C(t_0) = -(q_M(t) - q_M(t_0))$, the system can be reduced to a single first-order differential equation

$$\dot{\phi}_M = -\frac{1}{C}\left[\hat{q}_M(\phi_M) - \hat{q}_M(\phi_M(t_0)) - q_C(t_0)\right],$$

but still two initial conditions are needed to solve the latter.

6 Coupled Memristive Elements

During the construction of the input-state-output port-Hamiltonian formulation outlined in Section 4 it is implicitly assumed that there are no mutual dependencies between the memristive elements and no dependencies between the memristive elements and the environment. In order to account for such phenomena, the Dirac structure (16) is slightly extended as follows

$$\begin{aligned} \mathscr{D} = \Big\{ &(f_S, e_S, f_M, e_M, f_P, e_P) \in \mathscr{F} \times \mathscr{E} \,\big|\, f_S + Je_S + G_M f_M + G_P f_P = 0, \\ &-G_M^T e_S - G_{MM} f_M - G_{MP} f_P + e_M = 0, -G_P^T e_S + G_{MP}^T f_M - G_{PP} f_P + e_P = 0 \Big\}, \end{aligned} \tag{30}$$

where J, G_M, and G_P are as before, G_{MM} and G_{PP} are skew-symmetric matrices associated to, respectively, the mutual dependencies between the memristive elements and the flows and efforts at the external port, and G_{MP} reflects the coupling between the memristive elements and the environment. Using the effort-causal representation

[3] It is assumed here that the system is already interconnected before $t = t_0$, i.e., there are no impacts.

of the memristive port (21), the dynamics on $\mathscr{D}$ are then expressed as

$$\begin{aligned} -\dot{x}+J\frac{\partial H}{\partial x}(x)-G_M M_e(x_e)\dot{x}_e+G_P f_P&=0,\\ -G_M^T\frac{\partial H}{\partial x}(x)+G_{MM}M_e(x_e)\dot{x}_e-G_{MP}f_P+\dot{x}_e&=0,\\ -G_P^T\frac{\partial H}{\partial x}(x)-G_{MP}^T M_e(x_e)\dot{x}_e-G_{PP}f_P+e_P&=0. \end{aligned} \tag{31}$$

Solving the second equation for $\dot{x}_e$ yields

$$\dot{x}_e=\big(I+G_{MM}M_e(x_e)\big)^{-1}\left(G_M^T\frac{\partial H}{\partial x}(x)+G_{MP}f_P\right), \tag{32}$$

under the assumption that $\det(I+G_{MM}M_e(x_e))\neq 0$. Substitution of the latter into the first equation of (31) and some rearrangement of the various terms yields

$$\dot{x}=\big(J-G_M N_e(x_e)G_M^T\big)\frac{\partial H}{\partial x}(x)+\big(G_P-G_M N_e(x_e)G_{MP}\big)f_P,$$

together with the output equation

$$e_P=\big(G_P^T+G_{MP}^T N_e(x_e)G_M^T\big)\frac{\partial H}{\partial x}(x)+\big(G_{PP}+G_{MP}^T N_e(x_e)G_{MP}\big)f_P,$$

where $N_e(x_e):=M_e(x_e)\big(I+G_{MM}M_e(x_e)\big)^{-1}$. Finally, by defining the memristance matrix $N(x_e):=G_M N_e(x_e)G_M^T$, together with the matrices $P(x_e):=G_M N_e(x_e)G_{MP}$ and $S(x_e):=G_{MP}^T N_e(x_e)G_{MP}$, we obtain

$$\begin{aligned} \dot{x}&=\big(J-N(x_e)\big)\frac{\partial H}{\partial x}(x)+\big(G_P-P(x_e)\big)f_P\\ e_P&=\big(G_P+P(x_e)\big)^T\frac{\partial H}{\partial x}(x)+\big(G_{PP}+S(x_e)\big)f_P, \end{aligned} \tag{33}$$

which, together with (32), constitutes an input-state-output port-Hamiltonian system with memristive dissipation and direct feedthrough. Note that if there are no dependencies between the memristive elements, i.e., if $G_{MM}=0$, then $N_e(x_e)\equiv M_e(x_e)$, and thus $N(x_e)\equiv M(x_e)$ as should be expected.

For the system of the form (33) the power-balance takes the form

$$\dot{H}(x)=-\left(\frac{\partial H}{\partial x}(x)\right)^T N(x_e)\frac{\partial H}{\partial x}(x)-f_P^T\big(G_{PP}+S(x_e)\big)^T f_P-2f_P^T P^T(x_e)\frac{\partial H}{\partial x}(x)+e_P^T f_P,$$

which, by noting that $N(x_e)$ can be decomposed into a symmetric and a skew-symmetric part, i.e.,

$$\begin{aligned} N^s(x_e)&:=\frac{1}{2}\big(N(x_e)+N^T(x_e)\big),\\ N^a(x_e)&:=\frac{1}{2}\big(N(x_e)-N^T(x_e)\big), \end{aligned}$$

and by recalling that $G_{PP}=-G_{PP}^T$ and $S(x_e)=S^T(x_e)$, can be reduced to

$$\dot{H}(x)=-\begin{pmatrix}\frac{\partial H}{\partial x}(x)\\ f_P\end{pmatrix}^T\begin{pmatrix}N^s(x_e) & P(x_e)\\ P^T(x_e) & S(x_e)\end{pmatrix}\begin{pmatrix}\frac{\partial H}{\partial x}(x)\\ f_P\end{pmatrix}+e_P^T f_P. \tag{34}$$

Based on the latter power-balance equation, we conclude that under the condition

$$\begin{pmatrix}N^s(x_e) & P(x_e)\\ P^T(x_e) & S(x_e)\end{pmatrix}\succeq 0, \tag{35}$$

for all $x_e\in\mathscr{X}_e$, the system (33) is again passive with respect to the supply rate $e_P^T f_P$ and storage function the Hamiltonian function.

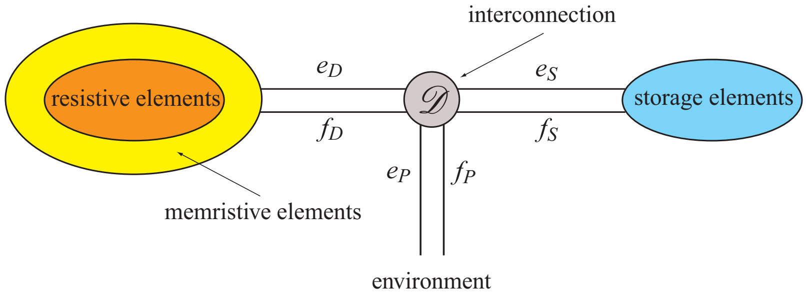

Figure 5: Port-Hamiltonian system with a single dissipative port containing memristors and linear resistors.

7 Final Remarks

In this paper, we have extended the existing port-Hamiltonian formalism with the inclusion of generalized memristive elements. Besides being a resistive element, a memristor also exhibits dynamics since the associated Ohmian laws are rather expressed in terms of differential equations. As a result, the state space manifold, as naturally defined by the storage elements, is augmented by the states associated to the memristive elements, and thus the order of complexity is, in general, defined by the total number of storage elements and memristors in the system. However, depending on the physical structure, there can exist constraints among some of the initial conditions. An example is provided by the mechanical system discussed in Subsection 5.2.

As briefly discussed in Subsection 4.3, the memristive port loses its relevance when the constitutive relationships of the memristive elements are linear. In such a case a memristor is equivalent to a linear resistor. Since in the port-Hamiltonian formalism usually only linear resistive elements are included, and memristors, like resistors, dissipate energy, we can combine both the resistive and memristive ports into a single *dissipative* port

$$\begin{aligned} f_D &= -D_e(x_e)e_D \\ e_D &= \dot{x}_e, \end{aligned}$$

where $(f_D, e_D) \in \mathscr{F}_D \times \mathscr{E}_D$ and $D_e(x_e)$ is referred to as the dissipation matrix. See also Figure 5.

As pointed out in [5, 10], memristors are just a special case of a much broader class of dynamical systems called memristive systems. The next step is to study under what conditions these systems can also be captured in the port-Hamiltonian formalism.

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8 Selected References

[1] P.C. Breedveld. Thermodynamic bond graphs and the problem of thermal inertance. *J. Franklin Inst.*, 314(1):15–40, July 1982.
[2] J. Cervera, A.J. van der Schaft, and A. Banos. Interconnections of port-Hamiltonian systems and composition of dirac structures. *Automatica*, 43:212–225, 2007.
[3] L.O. Chua. Memristor—the missing circuit element. *IEEE Trans. on Circ. Theory*, CT-18(2):507–519, September 1971.
[4] L.O. Chua. Nonlinear circuit foundations for nanodevices, part I: the four-element torus. *Proc. IEEE*, 91:1830–1859, November 2003.
[5] L.O. Chua and S.M. Kang. Memristive devices and systems. *Proc. IEEE*, 64:209–223, 1976.
[6] M. Dalsmo and A.J. van der Schaft. On representations and integrability of mathematical structures in energy-conserving physical systems. *SIAM J. of Contr. and Opt.*, 37:54–91, 1999.
[7] Y.F. Lam. Formulation of normal form equations of nonlinear networks containing memristors and coupled elements. *IEEE Trans. on Circ. Theory*, CT-19(6):585–594, November 1972.
[8] M.M. Milić and L.A. Novak. The anti-Lagrangian equations: a missing network description. *J. Franklin Inst.*, 307:183–191, 1979.
[9] G.F. Oster and D.M. Auslander. The memristor: A new bond graph element. *Trans. ASME on Dyn. Sys. Contr.*, 94(3):249–252, 1973.
[10] D.B. Strukov, G.S. Snider, D.R. Stewart, and R.S. Williams. The missing memristor found. *Nature*, 453:80–83, May 2008.
[11] A.J. Van der Schaft. *$\mathscr{L}_2$-Gain and Passivity Techniques in Nonlinear Control*. Springer-Verlag, 2000.