Modeling and Analysis of a Bicycle on the Three-Dimensional Space using the Projection Method

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Abstract. Our final goal of this study is to develop a bicycle riding support system. To develop the such system, because difficulties to ride bicycle are caused by its non linearity, it is important to consist of a strict nonlinear model of a bicycle. In this paper using the Projection Method and making out some appropriate constraint conditions, a way to derive a nonlinear bicycle model on the three-dimansional space is proposed. Some numerical simultions show the validity of the model.

1 Introduction

The automobile is one of the most familiar vehicle. But, in the recent years, the automobile has a lot of problems such as the greenhouse gases caused by automobiles, the cost of fuel up and so on. In this situation, riding a bicycle attracts attentions again as a no-emission vehicle. The bicycle also has many advantages such as keeping and increasing rider's health, relief of traffic congestion and energy efficiency. However, since the bicycle is an unstable system, a certain amount of skill is needed to perform stable riding. Especially, when the speed is low, its instability is increased. Therefore, riding support systems are needed for elderly people who can't treadle by an appropriate force at statring time, so the speed of the bicycle is low. Riding support systems are also needed for beginners for the same reason like as the elderly people. Many study on two-wheeled vehicles such as bicycles and electric motorbikes have been done[1][2]. Saguchi[3] has realized stable running on straight-line and curve motions using a model which is considered the skid of the wheels. Satou[4] has realized stabilizing a bicycle to control a handle and center of gravity(COG) by an attached cart-mass system. In these conventional studies, since only stabilizing at the upright position is considered, linear models that are linearized near the operating point is used. Especially, there is no literature using a model that is considered strict nonlinearity of the bicycle on the three-dimensional space, so it is difficult to consider stabilizing a bicycle when its speed is low. Therefore, our final goal of this study is to develop a stable bicycle riding support system using a strict nonlinear model of a bicycle.

In this paper, to consist of a strict nonlinear bicycle model, the Projection Method[5] is used. Each part of a wheel, a handle and a frame are modeled as subsystems, and these are connected by appropriate constraint forces that are not easy to derive. Validity of the model is shown using numerical simulations.

2 Modeling of a bicycle on the three-dimensional space

To derive a model of a bicycle, we assume that the equation of the motion of the wheel on the two-dimensional plane has already derived by the Projection Method. Fixing the two wheels and the handle by appropriate constraint conditions that are not easy to derive, the equation of the motion of the bicycle is derived by the Projection Method.

The model of the bicycle is shown in Figure 1 and Figure 2. Parameters of the bicycle is shown in Table 1.

2.1 The generalized coordinates, the generalized velocities and the transform matrix

The generalized coordinates x_a and the generalized velocities v_a are defined as follows:

$$\begin{aligned} x_a &= \begin{bmatrix} \theta_F & \phi_F & \psi_F & \theta_R & \phi_R & \psi_R & \theta_H & \phi_H & \psi_H & x_H & y_H & z_H \\ \theta_{FM} & \phi_{FM} & \psi_{FM} & x_{FM} & y_{FM} & z_{FM} \end{bmatrix}^T, \\ v_a &= \begin{bmatrix} \omega_{xR} & \omega_{yR} & \omega_{zH} & \omega_{xF} & \omega_{yF} & \omega_{zF} & \omega_{zR} & \omega_{xH} & \omega_{yH} & v_{xH} & v_{yH} & v_{zH} \\ \omega_{xFM} & \omega_{yFM} & \omega_{zFM} & v_{xFM} & v_{yFM} & v_{zFM} \end{bmatrix}^T. \end{aligned}$$

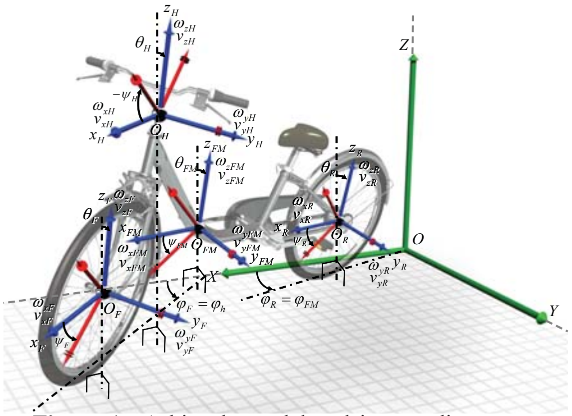

Figure 1: A bicycle model and its coordinate systems and parameters on 3D-space.

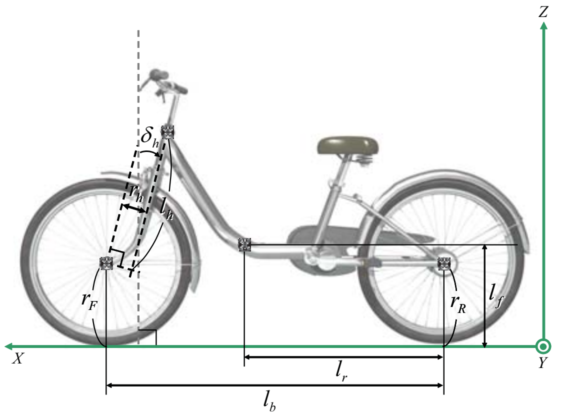

Figure 2: A bicycle model and its coordinate systems and parameters on the lateral plane.

Table 1: The parameter definition

Mass of the front wheel	m_F[kg]	1.5
Radius of the front wheel	r_F[m]	0.33
Mass of the rear wheel	m_R[kg]	1.5
Radius of the rear wheel	r_R[m]	0.33
Wheelbase	l_b[m]	1.07
Mass of the handle	m_H[kg]	1.0
Length of the front fork	l_h[m]	0.55
Angle of the front fork	δ_h[m]	20.0
Offset of the front wheel	r_h[m]	0.06
Mass of the frame	m_{FM}[kg]	16.0
Length from the rear wheel to the frame	l_b[m]	1.07
Height of the frame	l_f[m]	0.5
Inertia moment of the front wheel about x_F, z_F axis	I_{xzF} $[\mathrm{kg}\cdot\mathrm{m}^2]$	0.04
Inertia moment of the front wheel about y_F axis	I_{yF} $[\mathrm{kg}\cdot\mathrm{m}^2]$	0.08
Inertia moment of the rear wheel about x_R, z_R axis	I_{xzR} $[\mathrm{kg}\cdot\mathrm{m}^2]$	0.04
Inertia moment of the rear wheel about y_R axis	I_{yR} $[\mathrm{kg}\cdot\mathrm{m}^2]$	0.08
Inertia moment of the handle about x_H, z_H axis	I_{xzH} $[\mathrm{kg}\cdot\mathrm{m}^2]$	0.01
Inertia moment of the handle about y_H axis	I_{yH} $[\mathrm{kg}\cdot\mathrm{m}^2]$	0.01
Inertia moment of the frame about x_{FM}, z_{FM} axis	I_{xzFM} $[\mathrm{kg}\cdot\mathrm{m}^2]$	0.81
Inertia moment of the frame about y_{FM} axis	I_{yFM} $[\mathrm{kg}\cdot\mathrm{m}^2]$	1.62

Transformation matrix between x_a and v_a was represented as

$$A_a = \left[\begin{array}{cccccccccccccccccc}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -\tan\theta_F & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & (\cos\theta_F)^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & -\tan\theta_R & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & (\cos\theta_R)^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\tan\theta_H & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & (\cos\theta_H)^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cos\phi_H & -(\cos\theta_H\sin\phi_H) & \sin\phi_H\sin\theta_H & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sin\phi_H & \cos\phi_H\cos\theta_H & -(\cos\phi_H\sin\theta_H) & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sin\theta_H & \cos\theta_H & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\tan\theta_{FM} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (\cos\theta_{FM})^{-1} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cos\phi_{FM} & -(\cos\theta_{FM}\sin\phi_{FM}) & \sin\phi_{FM}\sin\theta_{FM} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sin\phi_{FM} & \cos\phi_{FM}\cos\theta_{FM} & -(\cos\phi_{FM}\sin\theta_{FM}) \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sin\theta_{FM} & \cos\theta_{FM}
\end{array}\right].$$

2.2 The generalized mass matrix and the generalized forces matrix

The dynamical system without any constraint can be derived using generalized coordinates. The equation of motion about the wheel is represented as

$$\begin{aligned}(I_w+m_w r_w{}^2)\dot{\omega}_{xw} =& g m_w r_w \sin\theta_w + (I_{yw}+m_w r_w{}^2)\omega_{yw}\omega_{zw} - I_{xzw}\tan\theta_w \omega_{zw}{}^2,\\ (I_{yw}+m_w r_w{}^2)\dot{\omega}_{yw} =& -m_w r_w{}^2\omega_{xw}\omega_{zw},\\ I_{xzw}\dot{\omega}_{zw} =& \omega_{zw}(-I_{yw}\omega_{yw}+I_{xzw}\tan\theta_w\omega_{zw}).\end{aligned} \tag{1}$$

The equation of motion about the handle and the frame are derived by Newton-Euler Method.

$$\begin{aligned}I_{xzH}\dot{\omega}_{xH} =& \omega_{zH}(-I_{xzH}\tan\theta_H\omega_{zH}+I_{yH}\omega_{yH})\\ I_{yH}\dot{\omega}_{yH} =& 0\\ I_{xzH}\dot{\omega}_{zH} =& \omega_{xH}(I_{xzH}\tan\theta_H\omega_{zH}+I_{yH}\omega_{yH})\\ m_H\dot{v}_{xH} =& m_H(v_{yH}-\tan\theta_H v_{zH})\omega_{zH}\\ m_H\dot{v}_{yH} =& -m_H(g\sin\theta_H - v_{zH}\omega_{xH}+v_{xH}\omega_{zH})\\ m_H\dot{v}_{zH} =& -m_H(g\cos\theta_H + v_{yH}\omega_{xH}-\tan\theta_H v_{xH}\omega_{zH})\\ I_{xzFM}\dot{\omega}_{xFM} =& \omega_{zFM}(-I_{xzFM}\tan\theta_{FM}\omega_{zFM}+I_{yFM}\omega_{yFM})\\ I_{yFM}\dot{\omega}_{yFM} =& 0\\ I_{xzFM}\dot{\omega}_{zFM} =& \omega_{xFM}(I_{xzFM}\tan\theta_{FM}\omega_{zFM}+I_{yFM}\omega_{yFM})\\ m_{FM}\dot{v}_{xFM} =& m_{FM}(v_{yFM}-\tan\theta_{FM}v_{zFM})\omega_{zFM}\\ m_{FM}\dot{v}_{yFM} =& -m_{FM}(g\sin\theta_{FM}-v_{zFM}\omega_{xFM}+v_{xFM}\omega_{zFM})\\ m_{FM}\dot{v}_{zFM} =& -m_{FM}(g\cos\theta_{FM}+v_{yFM}\omega_{xFM}-\tan\theta_{FM}v_{xFM}\omega_{zFM})\end{aligned} \tag{2}$$

Therefore, the generalized mass matrix M_a and the generalized forces matrix h_a of the bicycle without any constraint is represented as

$$\begin{aligned}M_a =& \mathrm{diag}(I_{xzR}+m_R r_R{}^2,\ I_{yR}+m_R r_R{}^2,\ I_{xzH},\ I_{xzF}+m_F r_F{}^2,\ I_{yF}+m_F r_F{}^2,\\ & I_{xzF}, I_{xzR},\ I_{xzH},\ I_{yH},\ m_H,\ m_H,\ m_H,\ I_{xzFM},\ I_{yFM},\ I_{xzFM},\ m_{FM},\ m_{FM},\ m_{FM}),\end{aligned}$$

$$h_a = \begin{bmatrix} g m_R r_R\sin\theta_R + (I_{yR}+m_R r_R{}^2)\omega_{yR}\omega_{zR} - I_{xzR}\tan\theta_R\omega_{zR}{}^2\\ -m_R r_R{}^2\omega_{xR}\omega_{zR}\\ \omega_{xH}(I_{xzH}\tan\theta_H\omega_{zH}-I_{yH}\omega_{yH})\\ g m_F r_F\sin\theta_F+(I_{yF}+m_F r_F{}^2)\omega_{yF}\omega_{zF}-I_{xzF}\tan\theta_F\omega_{zF}{}^2\\ -m_F r_F{}^2\omega_{xF}\omega_{zF}\\ \omega_{zF}(-I_{yF}\omega_{yF}+I_{xzF}\tan\theta_F\omega_{zF})\\ \omega_{zR}(-I_{yR}\omega_{yR}+I_{xzR}\tan\theta_R\omega_{zR})\\ \omega_{zH}(-I_{xzH}\tan\theta_H\omega_{zH}+I_{yH}\omega_{yH})\\ 0\\ m_H(v_{yH}-\tan\theta_H v_{zH})\omega_{zH}\\ -m_H(g\sin\theta_H-v_{zH}\omega_{xH}+v_{xH}\omega_{zH})\\ -m_H(g\cos\theta_H+v_{yH}\omega_{xH}-\tan\theta_H v_{xH}\omega_{zH})\\ \omega_{zFM}(-I_{xzFM}\tan\theta_{FM}\omega_{zFM}+I_{yFM}\omega_{yFM})\\ 0\\ \omega_{xFM}(I_{xzFM}\tan\theta_{FM}\omega_{zFM}-I_{yFM}\omega_{yFM})\\ m_{FM}(v_{yFM}-\tan\theta_{FM}v_{zFM})\omega_{zFM}\\ -m_{FM}(g\sin\theta_{FM}-v_{zFM}\omega_{xFM}+v_{xFM}\omega_{zFM})\\ -m_{FM}(g\cos\theta_{FM}+v_{yFM}\omega_{xFM}-\tan\theta_{FM}v_{xFM}\omega_{zFM}) \end{bmatrix}.$$

2.3 The constraint matrix

The constraint conditions between the front wheel and the rear wheel are held as follows:

- The relation between the arc by the front wheel and the arc by the rear wheel is

$$r_F\psi_F\cos(\phi_F-\phi_R) = r_R\psi_R. \tag{3}$$

- The relation between the arc by the front wheel and the track by the rear wheel is

$$r_F \psi_F \sin(\phi_F - \phi_R) = l_b \phi_R. \tag{4}$$

- The relation between inclination of the front wheel and the rear wheel is

$$r_R \omega_{xR} \cos\theta_R = r_F \omega_{xF} \cos\theta_F \cos(\phi_F - \phi_R). \tag{5}$$

The constraint conditions between the front wheel and the handle are held as follows:

- The positional constraint that the handle is connected to the front wheel is

$$x_H = R_z(\phi_H) R_x(\theta_H) R_y(\psi_H), \begin{bmatrix} -r_h \\ 0 \\ l_h \end{bmatrix} + x_F \tag{6}$$

where $x_F = \begin{bmatrix} x_F & y_F & z_F \end{bmatrix}^T$ is the vector of COG of the front wheel and, $x_H = \begin{bmatrix} x_H & y_H & z_H \end{bmatrix}^T$ is the vector of COG of the handle.The rotation matrix $R_z(\phi_H),\ R_x(\theta_H),\ R_y(\psi_H)$ are represented as

$$R_z(\phi_H) = \begin{bmatrix} \cos\phi_H & -\sin\phi_H & 0 \\ \sin\phi_H & \cos\phi_H & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{7}$$

$$R_x(\theta_H) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_H & -\sin\theta_H \\ 0 & \sin\theta_H & \cos\theta_H \end{bmatrix}, \tag{8}$$

$$R_y(\psi_H) = \begin{bmatrix} \cos\psi_H & 0 & \sin\psi_H \\ 0 & 1 & 0 \\ -\sin\psi_H & 0 & \cos\psi_H \end{bmatrix}. \tag{9}$$

- The relation between the angle of inclination of the front wheel and the angle of inclination of the handle, and the relation between the steering angle of the front wheel and the steering angle of the handle are

$$\theta_F = \theta_H, \tag{10}$$

$$\phi_F = \phi_H. \tag{11}$$

- The relation between the front fork angle and the handle angle is

$$\psi_H = -\delta_h. \tag{12}$$

The constraint conditions between the rear wheel and the frame are held as follows:

- The positional constraint that the frame is connected to the rear wheel is

$$x_{FM} = R_z(\phi_{FM}) R_x(\theta_{FM}) R_y(\psi_{FM}) \begin{bmatrix} l_r \\ 0 \\ l_f - r_R \end{bmatrix} + x_R, \tag{13}$$

where $x_R = \begin{bmatrix} x_R & y_R & z_R \end{bmatrix}^T$ is the vector of COG of the rear wheel and, $x_{FM} = \begin{bmatrix} x_{FM} & y_{FM} & z_{FM} \end{bmatrix}^T$ is the vector of COG of the frame. The rotation matrix $R_z(\phi_{FM}),\ R_x(\theta_{FM}),\ R_y(\psi_{FM})$ are represented as

$$R_z(\phi_{FM}) = \begin{bmatrix} \cos\phi_{FM} & -\sin\phi_{FM} & 0 \\ \sin\phi_{FM} & \cos\phi_{FM} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{14}$$

$$R_x(\theta_{FM}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_{FM} & -\sin\theta_{FM} \\ 0 & \sin\theta_{FM} & \cos\theta_{FM} \end{bmatrix}, \tag{15}$$

$$R_y(\psi_{FM}) = \begin{bmatrix} \cos\psi_{FM} & 0 & \sin\psi_{FM} \\ 0 & 1 & 0 \\ -\sin\psi_{FM} & 0 & \cos\psi_{FM} \end{bmatrix}. \tag{16}$$

- The relation between the angle of inclination of the rear wheel and the angle of inclination of the frame, and the relation between the steering angle of the rear wheel and the steering angle of the frame are

$$\theta_R = \theta_{FM}, \tag{17}$$

$$\phi_R = \phi_{FM}. \tag{18}$$

- The relation of the pitch angle of the frame is

$$\psi_{FM} = 0. \tag{19}$$

Using these constraints and the tangent velocity $\dot{q} = \begin{bmatrix} \omega_{xR} & \omega_{yR} & \omega_{zH} \end{bmatrix}^T$, the constraint matrix of the system is represented as

$$C_a = \begin{bmatrix} C_{a1} & C_{a2} \end{bmatrix}, \tag{20}$$

where the matrix C_{a1} that is multiplied by the tangent velocity of C_a is represented as

$$C_{a1} = \begin{bmatrix}
0 & r_R & 0 \\
0 & 0 & 0 \\
-r_R\cos\theta_R & 0 & 0 \\
0 & 0 & \alpha_{3,3} \\
0 & 0 & \alpha_{4,3} \\
0 & 0 & -\sin\theta_H(r_h\cos\psi_H - l_h\sin\psi_H) \\
0 & 0 & 0 \\
0 & 0 & \cos\theta_H \\
0 & 0 & -\tan\theta_H \\
r_R\cos\theta_R\sin\phi_R & r_R\cos\phi_R & 0 \\
-r_R\cos\theta_R\cos\phi_R & r_R\sin\phi_R & 0 \\
-r_R\sin\theta_R & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix},$$

and the matrix C_{a2} that is multiplied by vectors that consist of the other part of the tangent velocity is represented by

$$C_{a2} = \left[\begin{matrix}
0 & \alpha_{1,5} & \alpha_{1,6} & \alpha_{1,7} & 0 & 0 & 0 & 0 \\
0 & \alpha_{2,5} & \alpha_{2,6} & \alpha_{2,7} & 0 & 0 & 0 & 0 \\
\alpha_{3,4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\alpha_{4,4} & \alpha_{4,5} & 0 & 0 & \alpha_{4,8} & \alpha_{4,9} & \alpha_{4,10} & \alpha_{4,11} \\
\alpha_{5,4} & \alpha_{5,5} & 0 & 0 & \alpha_{5,8} & \alpha_{5,9} & \alpha_{5,10} & \alpha_{5,11} \\
\alpha_{6,4} & 0 & 0 & 0 & \alpha_{6,8} & \alpha_{6,9} & 0 & \alpha_{6,11} \\
-1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & -\cos\theta_F & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \cos\theta_R & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{matrix}\right.$$

$$\left.\begin{matrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\alpha_{4,12} & 0 & 0 & 0 & 0 & 0 & 0 \\
\alpha_{5,12} & 0 & 0 & 0 & 0 & 0 & 0 \\
\alpha_{6,12} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \alpha_{10,13} & \alpha_{10,14} & \alpha_{10,15} & \alpha_{10,16} & \alpha_{10,17} & \alpha_{10,18} \\
0 & \alpha_{11,13} & \alpha_{11,14} & \alpha_{11,15} & \alpha_{11,16} & \alpha_{11,17} & \alpha_{11,18} \\
0 & \alpha_{12,13} & \alpha_{12,14} & \alpha_{12,15} & 0 & \alpha_{12,17} & \alpha_{12,18} \\
0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\cos\theta_{FM} & 0 & 0 & 0 \\
0 & 0 & 1 & -\tan\theta_{FM} & 0 & 0 & 0
\end{matrix}\right],$$

where

$$\begin{aligned}
\alpha_{3,3} = & \cos\theta_H(\sin\phi_H(r_h\cos\psi_H - l_h\sin\psi_H) + \cos\phi_H\sin\theta_H(l_h\cos\psi_H + r_h\sin\psi_H)) \\
& - (\sin\theta_H\sin\phi_H(r_h\cos\psi_H - l_h\sin\psi_H) + \cos\phi_H(l_h\cos\psi_H + r_h\sin\psi_H))\tan\theta_H, \\
\alpha_{4,3} = & \cos\theta_H(\cos\phi_H(l_h\sin\psi_H - r_h\cos\psi_H) + \sin\theta_H\sin\phi_H(l_h\cos\psi_H + r_h\sin\psi_H)) \\
& - (\cos\phi_H\sin\theta_H(l_h\sin\psi_H - r_h\cos\psi_H) + \sin\phi_H(l_h\cos\psi_H + r_h\sin\psi_H))\tan\theta_H
\end{aligned}$$

$$
\begin{aligned}
\alpha_{1,5} =& -r_F\cos(\phi_F-\phi_R),\\
\alpha_{1,6} =& r_F\cos(\phi_F-\phi_R)\tan\theta_F + r_F\sec\theta_F\sin(\phi_F-\phi_R)\psi_F,\\
\alpha_{1,7} =& -r_R\tan\theta_R - r_F\sec\theta_R\sin(\phi_F-\phi_R)\psi_F,\\
\alpha_{2,5} =& -r_F\sin(\phi_F(t)-\phi_R),\\
\alpha_{2,6} =& r_F\sin(\phi_F-\phi_R)\tan\theta_F - r_F\cos(\phi_F-\phi_R)\sec\theta_F\psi_F,\\
\alpha_{2,7} =& \sec\theta_R(l_b + r_F\cos\phi_F - \phi_R\psi_F),\\
\alpha_{3,4} =& r_F\cos\theta_F\cos(\phi_F-\phi_R),\\
\alpha_{4,4} =& r_F\cos\theta_F\sin\phi_F,\\
\alpha_{4,5} =& r_F\cos\phi_F,\\
\alpha_{4,8} =& \cos\theta_H\sin\phi_H(l_h\cos\psi_H + r_h\sin\psi_H),\\
\alpha_{4,9} =& \sin\theta_H\sin\phi_H(r_h\cos\psi_H - l_h\sin\psi_H) + \cos\phi_H(l_h\cos\psi_H + r_h\sin\psi_H),\\
\alpha_{4,10} =& -\cos\phi_H,\\
\alpha_{4,11} =& \cos\theta_H\sin\phi_H,\\
\alpha_{4,12} =& -\sin\theta_H\sin\phi_H,\\
\alpha_{5,4} =& -r_F\cos\theta_F\cos\phi_F,\\
\alpha_{5,5} =& r_F\sin\phi_F,\\
\alpha_{5,8} =& -\cos\theta_H\cos\phi_H(l_h\cos\psi_H + r_h\sin\psi_H),\\
\alpha_{5,9} =& \cos\phi_H\sin\theta_H(l_h\sin\psi_H - r_h\cos\psi_H) + \sin\phi_H(l_h\cos\psi_H + r_h\sin\psi_H),\\
\alpha_{5,10} =& -\sin\phi_H,\\
\alpha_{5,11} =& -\cos\theta_H\cos\phi_H,\\
\alpha_{5,12} =& \cos\phi_H\sin\theta_H,\\
\alpha_{6,4} =& -r_F\sin\theta_F,\\
\alpha_{6,8} =& -\sin\theta_H(l_h\cos\psi_H + r_h\sin\psi_H),\\
\alpha_{6,9} =& \cos\theta_H(r_h\cos\psi_H - l_h\sin\psi_H),\\
\alpha_{6,11} =& -\sin\theta_H,\\
\alpha_{6,12} =& -\cos\theta_H,\\
\alpha_{10,13} =& \cos\theta_{FM}\sin\phi_{FM}((l_f-r_R)\cos\psi_{FM} - l_r\sin\psi_{FM}),\\
\alpha_{10,14} =& \cos\phi_{FM}((l_f-r_R)\cos\psi_{FM}),\\
& -l_r\sin\psi_{FM} - \sin\theta_{FM}\sin\phi_{FM}(l_r\cos\psi_{FM} + (l_f-r_R)\sin\psi_{FM}),\\
\alpha_{10,15} =& \cos\theta_{FM}(\cos\phi_{FM}\sin\theta_{FM}((l_f-r_R)\cos\psi_{FM} - l_r\sin\psi_{FM}),\\
& -\sin\phi_{FM}(l_r\cos\psi_{FM} + (l_f-r_R)\sin\psi_{FM})) - (\cos\phi_{FM}((l_f-r_R)\cos\psi_{FM},\\
& -l_r\sin\psi_{FM}) - \sin\theta_{FM}\sin\phi_{FM}(l_r\cos\psi_{FM} + (l_f-r_R)\sin\psi_{FM}))\tan\theta_{FM},\\
\alpha_{10,16} =& -\cos\phi_{FM},\\
\alpha_{10,17} =& \cos\theta_{FM}\sin\phi_{FM},\\
\alpha_{10,18} =& -\sin\theta_{FM}\sin\phi_{FM},\\
\alpha_{11,13} =& \cos\theta_{FM}\cos\phi_{FM}((r_R-l_f)\cos\psi_{FM} + l_r\sin\psi_{FM}),\\
\alpha_{11,14} =& \sin\phi_{FM}((l_f-r_R)\cos\psi_{FM} - l_r\sin\psi_{FM}) + \cos\phi_{FM}\sin\theta_{FM},\\
& (l_r\cos\psi_{FM} + (l_f-r_R)\sin\psi_{FM}),\\
\alpha_{11,15} =& \cos\theta_{FM}(\sin\theta_{FM}\sin\phi_{FM}((l_f-r_R)\cos\psi_{FM} - l_r\sin\psi_{FM}),\\
& +\cos\phi_{FM}(l_r\cos\psi_{FM} + (l_f-r_R)\sin\psi_{FM})) - (\sin\phi_{FM}((l_f-r_R)\cos\psi_{FM},\\
& -l_r\sin\psi_{FM}) + \cos\phi_{FM}\sin\theta_{FM}(l_r\cos\psi_{FM} + (l_f-r_R)\sin\psi_{FM}))\tan\theta_{FM},\\
\alpha_{11,16} =& -\sin\phi_{FM},\\
\alpha_{11,17} =& -\cos\theta_{FM}\cos\phi_{FM},\\
\alpha_{11,18} =& \cos\phi_{FM}\sin\theta_{FM},\\
\alpha_{12,13} =& \sin\theta_{FM}((r_R-l_f)\cos\psi_{FM} + l_r\sin\psi_{FM}),\\
\alpha_{12,14} =& -\cos\theta_{FM}(l_r\cos\psi_{FM} + (l_f-r_R)\sin\psi_{FM}),\\
\alpha_{12,15} =& \sin\theta_{FM}l_r\cos\psi_{FM} + l_f - r_R\sin\psi_{FM},\\
\alpha_{12,17} =& -\sin\theta_{FM},\\
\alpha_{12,18} =& -\cos\theta_{FM}.
\end{aligned}
$$

2.4 The orthogonal complement matrix to the constraint matrix

The generalized velocity v_a is devided into tangent velocity $\dot{q}$ and the other $\tilde{v}$. Because $C_a v_a = 0$,

$$C_a v_a = \begin{bmatrix} C_{a1} & C_{a2} \end{bmatrix} \begin{bmatrix} \dot{q} \\ \tilde{v} \end{bmatrix} = C_{a1}\dot{q} + C_{a2}\tilde{v} = 0.$$

Therefore,

$$\tilde{v} = -C_{a2}{}^{-1} C_1 \dot{q}.$$

Hence, the orthogonal complement matrix D_a satisfies $C_a D_a = 0$ and $C_a v_a = 0$ is represented by

$$D_a = \begin{bmatrix} I \\ -C_{a2}{}^{-1} C_{a1} \end{bmatrix}. \tag{21}$$

2.5 Dynamical system with constraints

Dynamical system of the constrained system with the constraint reaction forces $C_a^T \lambda$ are derived as follows:

$$M_a \dot{v}_a = h_a + C_a^T \lambda, \tag{22}$$

where λ is the Lagrange's multipliers. Eliminating λ from (22), the motion equation of the bicycle is derived as

$$D_a^T M_a D_a \ddot{q} + D_a^T M_a \dot{D}_a \dot{q} = D_a^T h_a.$$

3 Model verification

The nonlinear model that is derived in the previous section is verified to compare with behaviors of the real system and numerical simulations using it. The initial velocity of the rear wheel is 0.0 $[rad/s]$. Trajectories of the steering angle ϕ_h and the inclination angle of the frame θ_{FM} are shown in Figure 3. In the results, initial velocity of the rear wheel is set to 0.0 $[rad/s]$ and initial inclination of the frame is set to 0 $[deg]$. From Figure 3, It is shown that the steering angle and the inclination angle of the real system and these of the nonlinear model are not same. Because it is difficult to set the initial steering angle of the real system at $\phi_h = 0$, in fact, the initial steering angle is 0.4 $[deg]$. Therefore, we simulate two cases. One is the initial steering angle is set to 0.4 $[deg]$, and the other one is the initial steering angle is set to 30 $[deg]$. The results are shown in Figure 4 and Figure 5.

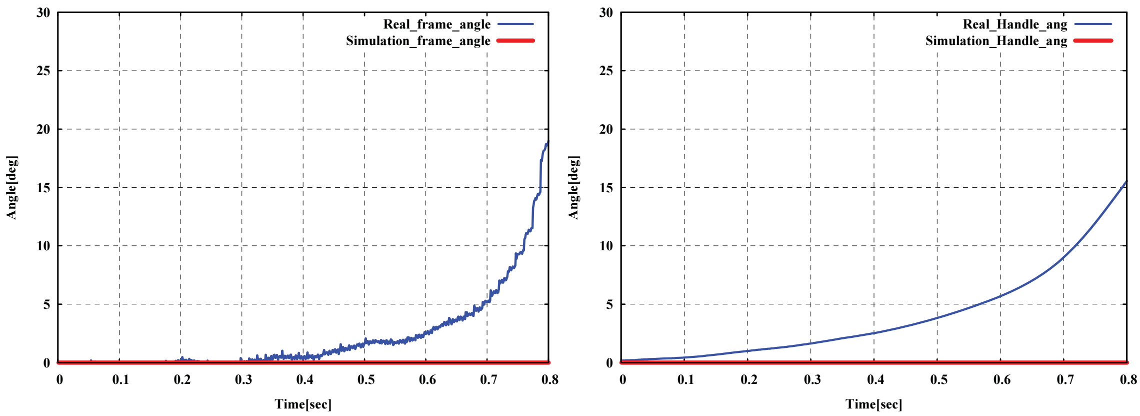

Figure 3: Trajectories of the steering angle ϕ_h and the inclination angle of the frame θ_{FM} when the initial inclination of the frame is set to 0 $[deg]$.

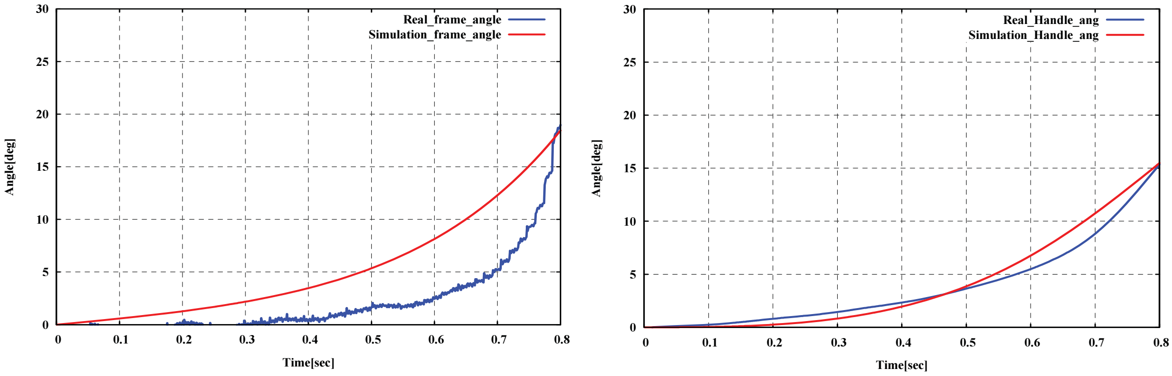

Figure 4: Trajectories of the steering angle ϕ_h and the inclination angle of the frame θ_{FM} when the initial inclination of the frame is set to 0.4 $[deg]$.

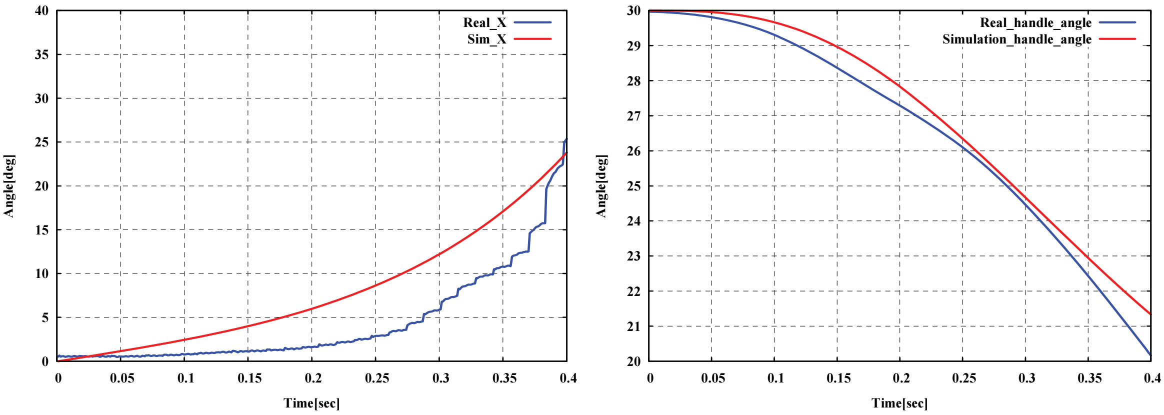

Figure 5: Trajectories of the steering angle ϕ_h and the inclination angle of the frame θ_{FM} when the initial inclination of the frame is set to 30 $[deg]$.

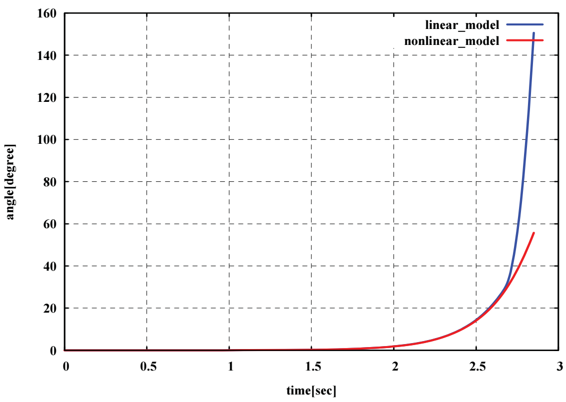

Figure 6: The trajectorys of the inclination angle of the frame for the proposed nonlinear model and the linear model.

From Figure 4, the handle rotates towards the angle which the frame is inclined when the frame is inclined. From Figure 5, the handle rotates towards the angle which the frame is not inclined when the frame is inclined. From Figure 4, and Figure 5, the behaviors of the proposed nonlinear model is similar to the real system indies.

To show the validity of the proposed nonlinear model, it is compared with the linear model by the simulation. The trajectries of the inclination angle of the frame for both models is shown in Figure 6. From Figure 6, it is clearly shown that the inclination angle of the linear model is deffer from the angle of the nonlinear model after $\theta_{FM} = 30$ $[deg]$, so the conventional linear models is not so useful to consider stablizing the bicycle in the wide inclination angles. Therefore, the nonlinear model is very important to develop a stable bicycle riding support system and analyze the stability of the bicycle under various situations.

4 Conclusion and future work

In this paper using the Projection Method and making out some appropriate constraint conditions, a way to derive a nonlinear bicycle model on the three-dimansional space is proposed. To show the validity of the nonlinear bicycle model, some numerical simulations are done. The simulation results also show differences between the nonlinear model and the conventional linear models that is very important to develop a stable bicycle riding support system and analyze the stability of the bicycle under various situations. To develop a more accurate nonlinear model, it is conncected a saddle, pedals and crank mechanisms to the proposed model using the Projection Method and some appropriate constraint conditions for the new attached parts, is a future work. The accurate nonlinear model helps to consider human inputs and to develop a stable bicycle riding support system.

5 References

[1] Åström,K.J, Klein,R.E, Lennartsson,A, Bicycle dynamics and control, Control Systems Magazine, IEEE, Volume 25, pp. 26-pp. 47, 2005

[2] Ko Iuchi, Hiroshi Niki and Toshiyuki Murakami, "Attitude Control of Bicycle Motion by Steering Angle and Variable COG Control", *IECON2005*, 2005

[3] SAGUCHI Taichi, YOSHIDA Kazuo, TAKAHASHI Masaki, Stable Running Control of Autonomous Bicycle Robot, Transactions of the Japan Society of Mechanical Engineers. C 2006, 2006

[4] Takushi Satou, Tohru Namekawa, Robust running stabilization of an independence running two-wheeled vehicle to change of speed and mass, The Society of Instrument and Control Engineers, Volume42, No. 1 1/8, 2006

[5] W. Arczewski, W.Blajer, " A Unified Approach to the Modelling of Holonomic and Nonholonomic Mechanical Systems ", Mathematical Modelling of Systems, 1996, Vol. 2, No. 3, pp157-pp174, 1996