APPROXIMATION OF A DISSIPATIVE SYSTEM WITH DIRAC STRUCTURE CONSERVATION: CASE STUDY

C. Chera[1], A. Nakrachi[2], G. Dauphin-Tanguy[1]
[1]Ecole Centrale de Lille, France [2]Lille University of Science and Technology, France
Laboratoire d'Automatique, Genie Informatique et Signal, CNRS UMR 8146

Corresponding author: C. Chera, Ecole Centrale de Lille, 59655 Villeneuve d'Ascq, Cité Scientifique, BP 48, France; kerac2002@yahoo.com

Abstract. In this paper, one shows that in a distributed parameter system represented in a port-Hamiltonian form we preserve the Dirac structure after a space discretization. Indeed, the conservation of the Dirac structure was shown only in the conservative case, without dissipation. The study is applied to the case of a transmission line represented by the telegrapher's equations.

1 Introduction

In the literature there are some papers where it is shown how port based network modelling of lumped and distributed parameter physical systems naturally leads to a geometrically defined class of systems, called port-Hamiltonian systems [8] [9]. The Hamiltonian approach starts from the principle of least action, uses the Euler-Lagrange equations and the Legendre transformation, and arrives to the Hamiltonian equation of motion.

In this approach the system behaviour is defined by a Dirac structure, which represents the power-conserving interconnection structure of the system, and the Hamiltonian, which is given by the total energy of the energy storing elements in the system. We can add energy-dissipating elements by terminating some of the system ports with resistive elements.

From a simulation point of view this approach is important because it gives information about the energy function and other conserved quantities in the system, which preferably should be kept in simulation. But in the case of distributed parameter systems an important problem concerns the incorporation of the numerical method, like finite elements and finite-difference, used for resolving the Partial Differential Equations (PDE), into this framework. We have an infinite-dimensional system that we need to approximate with a finite-dimensional one. The numerical methods for solving the PDE assume that the boundary conditions are given. In the case of telegrapher's equations the boundary conditions are the voltage and the current at both ends of the line. But these values cannot be considered given because the transmission line is connected to the other dynamic systems of the electrical network. So we need to approximate the distributed-parameter system with a finite dimensional system and to keep the power port structure.

In the case of the telegrapher's equation, the previous work have used an ideal model of the transmission line [3] [4]. In this paper we take a model that includes also the dissipative elements.

2 Full telegrapher's equation

We consider a transmission line with $\Omega = [0,1] \subset \mathbb{R}$ and define the energy variables as the charge density $q = q(t,x) \in \Lambda^1(\Omega)$ and the magnetic flux density $\varphi = \varphi(t,x) \in \Lambda^1(\Omega)$ where $\Lambda^1(\Omega)$ denotes the 1-forms space. The energy density (or the Hamiltonian density) h at time t in the homogeneous transmission line is given as:

$$H(q,\phi) = \frac{1}{2}\int_\Omega \left[q(t,x) \wedge * \frac{q(t,x)}{c} + \phi(t,x) \wedge * \frac{\phi(t,x)}{l} \right] \tag{1}$$

where l and c represent respectively inductance and capacitance density and where ($\wedge$) is the wedge product and (*) is the Hodge star operator, defined as

$$*(D) : \Lambda^k(D) \to \Lambda^{n-k}(D) \tag{2}$$

where D is an open set in Ω (a Riemannian manifold). In order to introduce the energy variables, we write the power balance of the transmission line starting with total energy

$$\frac{dH(q,\varphi)}{dt} = \int_\Omega \frac{\delta H(q,\varphi)}{\delta q} \wedge \frac{\partial q}{\partial t} + \frac{\delta H(q,\varphi)}{\delta \varphi} \wedge \frac{\partial \varphi}{\partial t} \tag{3}$$

where variational derivatives are given by

$$\frac{\delta H(q,\varphi)}{\delta q} = *\frac{1}{c} q(t,x) \qquad \frac{\delta H(q,\varphi)}{\delta \varphi} = *\frac{1}{l}\varphi(t,x) \tag{4}$$

We introduce the conjugate energy variables flow (1-form) and effort (0-forms) as follows:

$$\begin{cases} f_q(t,x) = \frac{\partial q(t,x)}{\partial t} & f_\varphi(t,x) = \frac{\partial \varphi(t,x)}{\partial t} \\ e_q(t,x) = *\frac{\delta H(q,\varphi)}{\delta q} & e_\varphi(t,x) = *\frac{\delta H(q,\varphi)}{\delta \varphi} \end{cases} \tag{5}$$

The equation (3) becomes,

$$\frac{dH(q,\varphi)}{dt} = \int_\Omega e_q \wedge f_q + e_\varphi \wedge f_\varphi \tag{6}$$

In addition, we have the telegraph equations written in terms of conjugate variables and differential forms:

$$\begin{cases} f_\varphi = -de_q - r(*e_\varphi) \\ f_q = -de_\varphi - g(*e_q) \end{cases} \tag{7}$$

where r and g represent respectively resistance and conductance density and where d is the usual exterior-derivative. Substitution in the equation (3) gives

$$\begin{aligned} \frac{dH}{dt} &= \int_\Omega e_q \wedge \left(-de_\varphi - g(*e_q)\right) + e_\varphi \wedge \left(-de_q - r(*e_\varphi)\right) \\ &= \int_\Omega \left[-e_q \wedge de_\varphi - e_\varphi \wedge de_q\right] - \int_\Omega \left[e_q \wedge g(*e_q) + e_\varphi \wedge r(*e_\varphi)\right] \\ &= -\int_\Omega d\left(e_\varphi \wedge e_q\right) - \int_\Omega \left[e_q \wedge g(*e_q) + e_\varphi \wedge r(*e_\varphi)\right] \\ &= -\int_{\partial\Omega} e_q \wedge e_\varphi - \int_\Omega \left(e_q \wedge g(*e_q) + e_\varphi \wedge r(*e_\varphi)\right) \end{aligned} \tag{8}$$

where $\partial\Omega = \{0,1\}$ represents boundary set of Ω. We have the following structure:

$$\int_\Omega e_q \wedge f_q + e_\varphi \wedge f_\varphi + \int_{\partial\Omega} e_b \wedge f_b + \int_\Omega e_{dq} \wedge f_{dq} + e_{d\varphi} \wedge f_{d\varphi} = 0 \tag{9}$$

where $e_b = e_q\big|_{\partial\Omega}$ and $f_b = e_\varphi\big|_{\partial\Omega}$ define the restriction of flux and effort on $\partial\Omega$,
$f_{d\varphi} = r(*e_\varphi) = \frac{r}{l}\varphi(t,x) \in \Lambda^1(\Omega)$, $f_{dq} = g(*e_q) = \frac{g}{c}q(t,x) \in \Lambda^1(\Omega)$ and
$e_{dq} = e_q$, $e_{d\varphi} = e_\varphi$.
The last two terms on the left side of equation (9) represent the power flow at the boundary and the dissipation power in a transmission line. In this way, the equation (7) becomes

$$\begin{aligned} f_\varphi &= -de_q - f_{d\varphi} \\ f_q &= -de_\varphi - f_{dq} \end{aligned} \tag{10}$$

and resulting port-Hamiltonian dissipative system:

$$\begin{pmatrix} f_\varphi \\ f_q \end{pmatrix} = \begin{pmatrix} 0 & -d \\ -d & 0 \end{pmatrix} \begin{pmatrix} e_\varphi \\ e_q \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} f_{d\varphi} \\ f_{dq} \end{pmatrix} \tag{11}$$

$$\begin{pmatrix} f_{d\varphi} \\ f_{dq} \end{pmatrix} = \begin{pmatrix} r* & 0 \\ 0 & g* \end{pmatrix} \begin{pmatrix} e_\varphi \\ e_q \end{pmatrix} \tag{12}$$

$$\begin{pmatrix} e_{b0} \\ e_{b1} \\ f_{b0} \\ f_{b1} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e_q(t,0) \\ e_q(t,1) \\ e_\varphi(t,0) \\ e_\varphi(t,1) \end{pmatrix} \tag{13}$$

b0 and *b1* denotes, respectively, the left and right boundary.
Note : In terms of current and voltage, the transmission line with dissipation satisfies the equation

$$\frac{dH}{dt} = v(t,0)i(t,0) - v(t,1)i(t,1) - \int_\Omega \left[ri^2(t,x) + gv^2(t,x)\right] \tag{14}$$

where $i(t,x) = *\frac{1}{l}\varphi(t,x)$ and $v(t,x) = *\frac{1}{c}q(t,x)$.

3 Spatial discretization

We will carry out a separation of variables and we will use the Whitney forms which make it possible to preserve the properties of the p-forms at the time of a spatial discretization [1].

Next we make the spatial discretization of the telegrapher's equations. The transmission line is split into m cells. Due to spatial compositionality (i.e. interconnection of two transmission lines via a common boundary once again gives a transmission line), we need to perform discretization to only one cell. That is to say the cell delimited by space $\Omega_c = [\alpha, \beta]$ One considers a cell (with the length $(\beta - \alpha)$), and we denote the spatial manifold $\Omega_c = [\alpha, \beta]$

We express the boundary variables as functions of the efforts:

$$\begin{aligned} e_{b\alpha}(t) &= e_q(t,\alpha) \quad e_{b\beta}(t) = e_q(t,\beta) \\ f_{b\alpha}(t) &= e_\varphi(t,\alpha) \quad f_{b\beta}(t) = e_\varphi(t,\beta) \end{aligned} \tag{15}$$

We consider the size of a sufficiently small cell, to be able to make the following approximations to represent flows inside the cell:

$$\begin{aligned} f_q(t,x) &\approx f_q^{\alpha\beta}(t) \cdot {}^1w_q(x) \\ f_\varphi(t,x) &\approx f_\varphi^{\alpha\beta}(t) \cdot {}^1w_\varphi(x) \end{aligned} \tag{16}$$

where ${}^1w_q, {}^1w_\varphi \in \Lambda^1(\Omega_c)$ are the 1-form satisfying the conditions:

$$\int_{\Omega_c} {}^1w_q(x) = 1 \quad \text{and} \quad \int_{\Omega_c} {}^1w_\varphi(x) = 1 \tag{17}$$

In the same way, the efforts $e_q(t,x)$ and $e_\varphi(t,x)$, inside the cell, are approximated by :

$$\begin{aligned} e_q(t,x) &= e_q^\alpha(t) \cdot w_q^\alpha(x) + e_q^\beta(t) \cdot w_q^\beta(x) \\ e_\varphi(t,x) &= e_\varphi^\alpha(t) \cdot w_\varphi^\alpha(x) + e_\varphi^\beta(t) \cdot w_\varphi^\beta(x) \end{aligned} \tag{18}$$

where $w_q^\alpha, w_q^\beta, w_\varphi^\alpha, w_\varphi^\beta \in \Lambda^0(\Omega_c)$ are the 0-forms satisfying the conditions:

$$\begin{aligned} w_q^\alpha(\alpha) = 1 \;\; w_q^\alpha(\beta) = 0 \quad w_q^\beta(\alpha) = 0 \quad w_q^\beta(\beta) = 1 \\ w_\varphi^\alpha(\alpha) = 1 \;\; w_\varphi^\alpha(\beta) = 0 \quad w_\varphi^\beta(\alpha) = 0 \quad w_\varphi^\beta(\beta) = 1 \end{aligned} \tag{19}$$

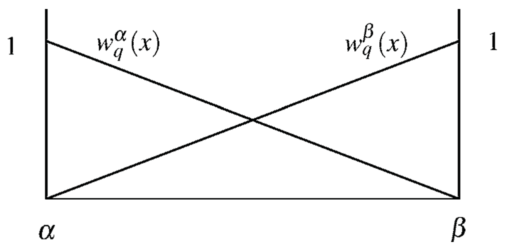

Figure 1: A Whitney 0-form

This Whitney 0-form, as shown figure (1), makes it possible to have the following relations

$$w_q^\alpha(x) + w_q^\beta(x) = 1 \quad \text{and} \quad w_\varphi^\alpha(x) + w_\varphi^\beta(x) = 1 \tag{20}$$

By substitution of (16) and (18) in (10) and by taking into account (12) we obtain:

$$\begin{aligned} f_\varphi^{\alpha\beta}(t)^1 w_\varphi(x) &= -e_q^\alpha(t) dw_q^\alpha(x) - e_q^\beta(t) dw_q^\beta(x) - r\left(e_\varphi^\alpha(*w_\varphi^\alpha) + e_\varphi^\beta(*w_\varphi^\beta)\right) \\ f_q^{\alpha\beta}(t)^1 w_q(x) &= -e_\varphi^\alpha(t) dw_\varphi^\alpha(x) - e_\varphi^\beta(t) dw_\varphi^\beta(x) - g\left(e_q^\alpha(*w_q^\alpha) + e_q^\beta(*w_q^\beta)\right) \end{aligned} \tag{21}$$

In order to determine the dynamic equations of inside a cell, we integrate in space the equations above:

$$\begin{aligned} f_\varphi^{\alpha\beta}(t) \int_{\Omega_c} {}^1w_\varphi(x) &= -e_q^\alpha(t) \int_{\Omega_c} dw_q^\alpha(x) - e_q^\beta(t) \int_{\Omega_c} dw_q^\beta(x) - re_\varphi^\alpha \int_{\Omega c} (*w_\varphi^\alpha) - re_\varphi^\beta \int_{\Omega_c} (*w_\varphi^\beta) \\ f_q^{\alpha\beta}(t) \int_{\Omega_c} {}^1w_q(x) &= -e_\varphi^\alpha(t) \int_{\Omega_c} dw_\varphi^\alpha(x) - e_\varphi^\beta(t) \int_{\Omega_c} dw_\varphi^\beta(x) - ge_q^\alpha \int_{\Omega c} (*w_q^\alpha) - ge_q^\beta \int_{\Omega_c} (*w_q^\beta) \end{aligned} \tag{22}$$

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Using the relations (17) and the properties

$$\int_{\Omega} dw = \int_{\partial\Omega} w \quad \text{and} \quad (*w) = wdx \tag{23}$$

where w is a 0-form and $(*w)$ is a 1-form,

$$w^{\alpha}_{\varphi} = w^{\alpha}_{q} = \frac{x-\beta}{\alpha-\beta} \text{ and } w^{\beta}_{\varphi} = w^{\beta}_{q} = \frac{x-\alpha}{\beta-\alpha} \tag{24}$$

the Whitney 0-form leads to the following relations:

$$\begin{array}{l} f^{\alpha\beta}_{\varphi}(t) = e^{\alpha}_{q}(t) - e^{\beta}_{q}(t) - \frac{1}{2}r_{\alpha\beta}(e^{\alpha}_{\varphi}(t) + e^{\beta}_{\varphi}(t)) \\ f^{\alpha\beta}_{q}(t) = e^{\alpha}_{\varphi}(t) - e^{\beta}_{\varphi}(t) - \frac{1}{2}g_{\alpha\beta}(e^{\alpha}_{q}(t) + e^{\beta}_{q}(t)) \end{array} \tag{25}$$

where $r_{\alpha\beta} = r(\beta-\alpha)$ and $g_{\alpha\beta} = g(\beta-\alpha)$

We arrive to the following spatial discretization representation of this typical cell

$$\begin{pmatrix} f_{b\alpha}(t) \\ f_{b\beta}(t) \\ e_{b\alpha}(t) \\ e_{b\beta}(t) \\ f^{\alpha\beta}_{\varphi}(t) \\ f^{\alpha\beta}_{q}(t) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -.5r_{\alpha\beta} & -.5r_{\alpha\beta} & 1 & -1 \\ 1 & -1 & -.5g_{\alpha\beta} & -.5g_{\alpha\beta} \end{pmatrix} \begin{pmatrix} e^{\alpha}_{\varphi}(t) \\ e^{\beta}_{\varphi}(t) \\ e^{\alpha}_{q}(t) \\ e^{\beta}_{q}(t) \end{pmatrix} \tag{26}$$

It remains to check that this is a port Hamiltonian system corresponding to a cell which preserves the structure of Dirac. This corresponds to an instantaneous conservation of the power (power net).

$$P_{\Omega_c} = \int_{\Omega_c} e_q \wedge f_q + \int_{\Omega_c} e_{\varphi} \wedge f_{\varphi} + \int_{\partial\Omega_c} e_b \wedge f_b \tag{27}$$

One replaces $e_q(t,x)$, $e_{\varphi}(t,x)$, $f_q(t,x)$ and $f_{\varphi}(t,x)$ by their approximations given in equations (16) and (18) to obtain the following expression:

$$P_{\Omega_c} = \int_{\Omega_c} \left(e^{\alpha}_{q} w^{\alpha}_{q} + e^{\beta}_{q} w^{\beta}_{q} \right) f^{\alpha\beta}_{q} {}^{1}w_q + \int_{\Omega_c} \left(e^{\alpha}_{\varphi} w^{\alpha}_{\varphi} + e^{\beta}_{\varphi} w^{\beta}_{\varphi} \right) f^{\alpha\beta}_{\varphi} {}^{1}w_{\varphi} + e_{b\beta} f_{b\beta} - e_{b\alpha} f_{b\alpha} \tag{28}$$

or

$$P_{\Omega_c} = \left(e^{\alpha}_{q} \int_{\Omega_c} w^{\alpha}_{q} {}^{1}w_q + e^{\beta}_{q} \int_{\Omega_c} w^{\beta}_{q} {}^{1}w_q \right) f^{\alpha\beta}_{q} + \left(e^{\alpha}_{\varphi} \int_{\Omega_c} w^{\alpha}_{\varphi} {}^{1}w_{\varphi} + e^{\beta}_{\varphi} \int_{\Omega_c} w^{\beta}_{\varphi} {}^{1}w_{\varphi} \right) f^{\alpha\beta}_{\varphi} + e_{b\beta} f_{b\beta} - e_{b\alpha} f_{b\alpha} \tag{29}$$

Before developing calculations, we establish initially some relations between the various 1-forms brought into play.

Combination of the equations (21) and (25) gives

$$\begin{array}{l} (e^{\alpha}_{q} - e^{\beta}_{q} - \frac{1}{2}r_{\alpha\beta}(e^{\alpha}_{\varphi} + e^{\beta}_{\varphi}))^{1}w_{\varphi} = -e^{\alpha}_{q} dw^{\alpha}_{q} - e^{\beta}_{q} dw^{\beta}_{q} - r\left(e^{\alpha}_{\varphi}(*w^{\alpha}_{\varphi}) + e^{\beta}_{\varphi}(*w^{\alpha}_{\varphi}) \right) \\ (e^{\alpha}_{\varphi} - e^{\beta}_{\varphi} - \frac{1}{2}g_{\alpha\beta}(e^{\alpha}_{q} + e^{\beta}_{q}))^{1}w_{q} = -e^{\alpha}_{\varphi} dw^{\alpha}_{\varphi} - e^{\beta}_{\varphi} dw^{\beta}_{\varphi} - g\left(e^{\alpha}_{q}(*w^{\alpha}_{q}) + e^{\beta}_{q}(*w^{\alpha}_{q}) \right) \end{array} \tag{30}$$

From them results:

$$\begin{array}{l} {}^{1}w_{\varphi} = -dw^{\alpha}_{q} = dw^{\beta}_{q} \\ {}^{1}w_{q} = -dw^{\alpha}_{\varphi} = dw^{\beta}_{\varphi} \end{array} \tag{31}$$

In addition, the use of the Whitney forms enables us to have the following relation:

$${}^{1}w_q = w^{\alpha}_{q} dw^{\beta}_{q} - w^{\beta}_{q} dw^{\alpha}_{q} = w^{\alpha}_{q} {}^{1}w_{\varphi} + w^{\beta}_{q} {}^{1}w_{\varphi} = (w^{\alpha}_{q} + w^{\beta}_{q}) {}^{1}w_{\varphi} = {}^{1}w_{\varphi} \tag{32}$$

We take $\gamma = \int_{\Omega_c} w^{\alpha}_{q} {}^{1}w_q$, which leads to

$$\int_{\Omega_c} w^{\beta}_{q} {}^{1}w_q = 1 - \gamma$$

$$\int_{\Omega_c} w^{\alpha}_{\varphi} {}^{1}w_{\varphi} = 1 - \gamma$$

$$\int_{\Omega_c} w^{\beta}_{\varphi} {}^{1}w_{\varphi} = \gamma$$

Indeed

$$\int_{\Omega_c} w_q^{\alpha 1} w_q + \int_{\Omega_c} w_\varphi^{\alpha 1} w_\varphi = -\int_{\Omega_c} w_q^\alpha dw_\varphi^\alpha - \int_{\Omega_c} w_\varphi^\alpha dw_q^\alpha = -\int_{\Omega_c} d(w_q^\alpha w_\varphi^\alpha) = -\left(w_q^\alpha(\beta) w_\varphi^\alpha(\beta) - w_q^\alpha(\alpha) w_\varphi^\alpha(\alpha)\right) = 1 \tag{33}$$

The equation (29) becomes :

$$P_{\Omega_c} = \left(\gamma e_q^\alpha + (1-\gamma) e_q^\beta\right) f_q^{\alpha\beta} + \left((1-\gamma) e_\varphi^\alpha + \gamma e_\varphi^\beta\right) f_\varphi^{\alpha\beta} + e_{b\beta} f_{b\beta} - e_{b\alpha} f_{b\alpha} \tag{34}$$

We say that

$$\begin{array}{ll} e_{b\alpha} = e_q^\alpha & e_{b\beta} = e_q^\beta \\ f_{b\alpha} = e_\varphi^\alpha & f_{b\beta} = e_\varphi^\beta \end{array} \tag{35}$$

What makes it possible to write the equations (25) in the form:

$$\begin{array}{l} f_\varphi^{\alpha\beta}(t) = e_{b\alpha} - e_{b\beta} - \frac{1}{2} r_{\alpha\beta}(f_{b\alpha} + f_{b\beta}) \\ f_q^{\alpha\beta}(t) = f_{b\alpha} - f_{b\beta} - \frac{1}{2} g_{\alpha\beta}(e_{b\alpha} + e_{b\beta}) \end{array} \tag{36}$$

For the efforts of the cell:

$$e_q^{\alpha\beta} = \gamma e_{b\alpha} + (1-\gamma) e_{b\beta} \text{ et } e_\varphi^{\alpha\beta} = (1-\gamma) f_{b\alpha} + \gamma f_{b\beta} \tag{37}$$

the instantaneous power is written then:

$$P_{\Omega_c} = \left\langle e^{\alpha\beta} \middle| f^{\alpha\beta} \right\rangle = e_q^{\alpha\beta} f_q^{\alpha\beta} + e_\varphi^{\alpha\beta} f_\varphi^{\alpha\beta} + e_{b\beta} f_{b\beta} - e_{b\alpha} f_{b\alpha} \tag{38}$$

where

$$e^{\alpha\beta} = \begin{pmatrix} e_\varphi^{\alpha\beta} & e_q^{\alpha\beta} & e_{b\alpha} & e_{b\beta} \end{pmatrix}^t \qquad f^{\alpha\beta} = \begin{pmatrix} f_\varphi^{\alpha\beta} & f_q^{\alpha\beta} & f_{b\alpha} & f_{b\beta} \end{pmatrix} \tag{39}$$

From (35) and (36) it comes

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\gamma & \gamma-1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & .5g_{\alpha\beta} & .5g_{\alpha\beta} \end{pmatrix}}_{F^{\alpha\beta}} \begin{pmatrix} e_\varphi^{\alpha\beta} \\ e_q^{\alpha\beta} \\ e_{b\alpha} \\ e_{b\beta} \end{pmatrix} + \underbrace{\begin{pmatrix} 0 & 0 & \gamma-1 & -\gamma \\ 0 & 0 & 0 & 0 \\ 1 & 0 & .5r_{\alpha\beta} & .5r_{\alpha\beta} \\ 0 & 1 & -1 & 1 \end{pmatrix}}_{E^{\alpha\beta}} \begin{pmatrix} f_\varphi^{\alpha\beta} \\ f_q^{\alpha\beta} \\ f_{b\alpha} \\ f_{b\beta} \end{pmatrix} = 0 \tag{40}$$

With $\gamma = 1/2$ in the case of the approximations of Whitney for the 0-forms and the 1-forms. We denote the space of admissible efforts by e, and the domain of admissible flows by f, such that the following relation is satisfied by

$$D = \left\{(f,e) \in^4 : E^{\alpha\beta} e^{\alpha\beta} + F^{\alpha\beta} f^{\alpha\beta} = 0\right\} \tag{41}$$

D is a Dirac structure with respect to the bilinear form if and only if the following two conditions are satisfied

$$rank\left[E^{\alpha\beta} \quad F^{\alpha\beta}\right] = 4 \qquad E^{\alpha\beta}(F^{\alpha\beta})^t + F^{\alpha\beta}(E^{\alpha\beta})^t = 0 \tag{42}$$

After computation we show that this is true and the two conditions are satisfied [5][2].

4 Constitutive equations

To complete calculations, we will determine the expressions of the charge $q_{\alpha\beta}(t)$ and the magnetic flux $\varphi_{\alpha\beta}(t)$ and their variations on the cell level.

$$\begin{array}{l} \varphi(t,x) = \varphi_{\alpha\beta}(t)^1 w(x) \\ q(t,x) = q_{\alpha\beta}(t)^1 w(x) \end{array} \tag{43}$$

The total energy of the cell is given by

$$\begin{aligned} h_{\alpha\beta}(\varphi_{\alpha\beta}, q_{\alpha\beta}) &= \int_{\Omega c} \tfrac{1}{2c} q_{\alpha\beta}^1 w \wedge q_{\alpha\beta}(*^1 w) + \int_{\Omega c} \tfrac{1}{2l} \varphi_{\alpha\beta}^1 w \wedge \varphi_{\alpha\beta}(*^1 w) \\ &= \left(\frac{\varphi_{\alpha\beta}^2(t)}{2l} + \frac{q_{\alpha\beta}^2(t)}{2c}\right) \int_{\Omega c} {}^1 w \wedge (*^1 w) \\ &= \frac{\varphi_{\alpha\beta}^2(t)}{2l_{\alpha\beta}} + \frac{q_{\alpha\beta}^2(t)}{2c_{\alpha\beta}} \end{aligned} \tag{44}$$

with

$$l_{\alpha\beta} = \frac{l}{\int_{\Omega c} {}^1w \wedge (*{}^1w)} = l(\beta - \alpha) \tag{45}$$

and

$$c_{\alpha\beta} = \frac{c}{\int_{\Omega c} {}^1w \wedge (*{}^1w)} = c(\beta - \alpha) \tag{46}$$

In addition, we have the following bonds:

$$\frac{\partial q(t,x)}{\partial t} = \frac{dq_{\alpha\beta}}{dt} {}^1w(x) = f_q^{\alpha\beta}(t) {}^1w(x) \tag{47}$$

then

$$\begin{array}{ll} \frac{dq_{\alpha\beta}}{dt} = f_q^{\alpha\beta}(t) & e_q^{\alpha\beta}(t) = \frac{\partial h_{\alpha\beta}}{\partial q} = \frac{q_{\alpha\beta}}{c_{\alpha\beta}} \\ \frac{d\varphi_{\alpha\beta}}{dt} = f_\varphi^{\alpha\beta}(t) & e_\varphi^{\alpha\beta}(t) = \frac{\partial h_{\alpha\beta}}{\partial \varphi} = \frac{\varphi_{\alpha\beta}}{l_{\alpha\beta}} \end{array} \tag{48}$$

and from equation (36) we have

$$\begin{array}{l} e_q^{\alpha\beta} = \frac{1}{2}\left(e_{b\alpha} + e_{b\beta}\right) \\ e_\varphi^{\alpha\beta} = \frac{1}{2}\left(f_{b\alpha} + f_{b\beta}\right) \end{array} \tag{49}$$

The dynamics of the cell is given then by:

$$\begin{array}{l} \frac{d\varphi_{\alpha\beta}}{dt} = e_{b\alpha} - e_{b\beta} - r_{\alpha\beta}\frac{\varphi_{\alpha\beta}}{l_{\alpha\beta}} \\ \frac{dq_{\alpha\beta}}{dt} = f_{b\alpha} - f_{b\beta} - g_{\alpha\beta}\frac{q_{\alpha\beta}}{c_{\alpha\beta}} \end{array} \tag{50}$$

It is known that connection of two Dirac structures gives a Dirac structure. Thus the whole transmission line can be reconstructed by the connection of a fixed number of cells in advance.

5 Conclusion

In this paper, we have shown, in the case of a transmission line represented by the telegrapher's equations with dissipation that we can preserve the Dirac structure of a distributed parameter system represented in the form of a port-Hamiltonian, after a space discretization. This gives us the possibilities to develop a further study from a control point of view.

6 References

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