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DETERMINATION OF INITIAL AMOUNT OF MATERIAL OF INTERMEDIATE STORAGES BY DIFFERENCE EQUATIONS IN DISCRETE STOCHASTIC MODELS

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Abstract. In this paper a difference equation arising from process engineering application is presented and investigated. Starting out of a discrete stochastic model we introduce a sequence of probabilities which expresses reliabilities. Using theorems of probability theory we set up a difference equation for the sequence, we prove the existence and the uniqueness of the solution under certain conditions and in special cases we solve it. We apply the solutions to determinate appropriate initial amount of material.

1 Introduction

Intermediate storages play important role in process engineering systems. They connect process subsystems with different operational characteristics. One of the subsystems fill some material into the intermediate storage, the second subsystem withdraw the material and use it. It is an important question how much initial amount of material is needed to assure the continuous work of the destination system. In order to find the appropriate amount of material it is worth investigating the volume of amount of material in the function of the time.

In most of practical problems the process is rather stochastic than deterministic [3], hence the appropriate amount of material can be determine to a given reliability level. The amount of material in the storage describes a stochastic process and we would like determine the distribution of its minimum value. In batch/continuous systems the distribution function of the maximum value of material in the storage satisfies such integral equations which can be transformed into integro-differential or differential equation with advances in their arguments [4,5,7]. In this paper we investigate the problem in discrete stochastic model and we deal with the probabilities of running out of material. We mention that this model can be interpreted as a discrete version of a risk process, and the problem detailed corresponds to the level crossing problem in that interpretation [1,2].

The structure of the paper will be the following: first we present the problem and the model which will be investigated, we introduce notations. Then using the methods of probability theory we prove the difference equation for the probability of running out of material. We prove the existence and uniqueness of the solution under certain conditions dealing with the rate of convergence as well. In special cases we give explicit solutions for the equation. Using them we present two examples in which we determine the initial amount of material for a given reliability level.

2 The model

Let consider the following processing system. Some of batch units produce material and some of other units use them at different time. The amount of material produced is filled into the intermediate storage which stores it and the material is withdrawn from it at constant volumetric rate *c*. The filling time points are supposed to be random.

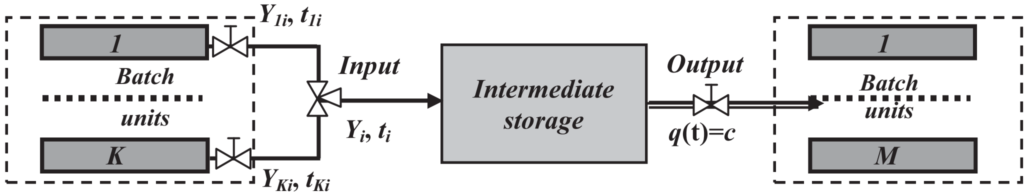

Figure 2.1. Intermediate storage connecting two batch subsystems of a processing system

Let $t_0 = 0$, and let us denote the times between the consecutive fillings by t_k (*k=1,2,3,...*), which are nonnegative, independent, identically distributed random variables. The counting process $\{N(t) : t \geq 0\}$ denotes the number of fillings up to time *t*, and is defined as

$$N(t)=\begin{cases}0, & \text{if } t_1>t\\ \max\left\{l:\sum_{i=1}^{l}t_i\le t\right\}, & \text{if } t_1\le t\end{cases}.$$

The amount of the k-th filling is denoted by Y_k, $k=1,2...$, and variables Y_k, $k=1,2...$ are also nonnegative, independent and identically distributed random variables. Also, we assume that $N(t)$ and $\{Y_k\}_{k=1}^{\infty}$ are independent.

The amount of material being in the intermediate storage can be expressed as

$$V(t)=z_0+\sum_{k=1}^{N(t)}Y_k-ct\,,\ t\geq 0 \tag{2.1}$$

where $z_0\ge 0$ is the initial amount of material. If we avoid running out of material, then the following inequality holds: $V(t)>0$ for every $t\ge 0$, running out of material means $V(t)<0$ for some $t\ge 0$.

If we investigate probability of running out of material we deal with the function

$$\psi_2(x)=P(x-ct+\sum_{k=1}^{N(t)}Y_k<0 \quad \text{for} \quad \text{some} \quad t\ge 0).$$

If distributions of random variables t_k and Y_k are continuous, the integral equation for $\psi_2(x)$ in special cases is proved in [3, 4] and concerning $R_2(x)=1-\psi_2(x)$ is presented and analyzed. In our previous publications we presented integro-differential equation satisfied by $R_2(x)$, we transformed the equations into integral or differential equation with advances in the argument and we solved them.

Often random variables Y_k, $k=1,2...$ and t_k, $k=1,2...$ have discrete distributions. In these cases the appropriate probabilities satisfies difference equations instead of integro-differential or differential equations. In this paper we suppose both the time intervals between consecutive filling times and the amount of material have discrete distributions with notation $P(t_k=j)=f(j)$ $j=0,1,2...$ and $P(Y_k=i)=g(i)$, $i=0,1,2,...$, furthermore $c=1$. Now $f(j)\ge 0$, $g(i)\ge 0$, $\sum_{j=0}^{\infty}f(j)=1$, $\sum_{i=0}^{\infty}g(i)=1$. We assume that expectations of the random variables are finite, that is $\mu_f=\sum_{j=0}^{\infty}jf(j)<\infty$, $\mu_g=\sum_{i=0}^{\infty}ig(i)<\infty$.

We note that $f(0)>0$ expresses that more than one filling can happen at the same time and $g(0)>0$ expresses that sometimes the amount of filled material can be zero (for example failure happens).

Let $x(n)$ be defined as the probability of running out of material, that is

$$x(n)=P(n+\sum_{k=1}^{N(m)}Y_k-m\le 0 \quad \text{for} \quad \text{some} \quad m=0,1,...)\,,\ n=0,1,2,... \tag{2.2}$$

the argument n expresses the initial amount of material.

We note that $x(0)=1$ obviously holds. Moreover $0\le x(n)\le 1$ and the sequence is monotone decreasing.

3 Difference equation for the sequence $x(n)$

Theorem 3.1

Sequence $x(n)$ $n=0,1,2...$ satisfies the following difference equation:

$$x(n)=\sum_{i=0}^{\infty}\sum_{j=0}^{n-1}x(n-j+i)f(j)g(i)+\sum_{j=n}^{\infty}f(j)\,. \tag{3.1}$$

Proof: Let $Y_1=j$ and $t_1=i$. We apply the theorem of total probability with conditions $Y_1=j$ and $t_1=i$ $j=0,1,...$, $i=0,1,2,...$

$$x(n)=P(m-\sum_{k=1}^{N(m)}Y_k-n\le 0 \quad \text{for} \quad \text{any} \quad m\in N)=$$

$$=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}P(m-\sum_{k=1}^{N(m)}Y_k-n\le 0 \quad \text{for} \quad \text{any} \quad m\in N\,|\,Y_1=i \quad \text{and} \quad t_1=j)f(j)g(i)\,. \tag{3.2}$$

If $j \ge n$, then at time point n we run out of material. If $j < n$ and the amount of the filled material is *i,* then the running out of material has not happened to the first filling, and process will be renewed. Amount of material being in the storage is $n-j+i$, this corresponds the initial amount of material if the process would begin at time point t_1. Hence (3.2) equals

$$\sum_{i=0}^{\infty}\sum_{j=0}^{n-1}P(m-\sum_{k=1}^{N(m)}Y_k+n<0 \quad \text{for} \quad \text{any} \quad m\in N\,|\,Y_1=i \quad \text{and} \quad t_1=j)f(j)g(i)+\sum_{j=n}^{\infty}f(j)$$

$$=\sum_{i=0}^{\infty}\sum_{j=1}^{n-1}x(n-j+i)f(j)g(i)+\sum_{j=n}^{\infty}f(j)\,.$$

We draw the attention that Eq. (3.1) is significantly different from the difference equation satisfied by the sequence of probabilities expressing overflow probabilities in sizing intermediate storage, presented in [8].

4 Existence and uniqueness of the solution of Equation (3.1)

Theorem 4.1

If $\mu_g > \mu_f$, and ${\tau_1}^{-n}\sum_{j=n}^{\infty}f(j)$ is bounded with some $0<\tau_1<1$ then there exists a τ_0, $0<\tau_1\le\tau_0<1$ for which the solution of Eq.(3.1) $x(n)$ with property $|x(n)|\le{\tau_0}^nK$, $K\in R$ is unique.

Proof: First let multiply Eq. (3.1) by τ^n with $\tau_1<\tau<1$.

Introduce notation $y(n)=\tau^n x(n)$. Sequence $y(n)$ $n=0,1,2...$ satisfies the following difference equation:

$$y(n)=\sum_{i=0}^{\infty}\sum_{j=0}^{n-1}y(n-j+i)f(j)g(i)\tau^{-j+i}+\tau^{-n}\sum_{j=n}^{\infty}f(j)\,. \tag{4.1}$$

Let us introduce the operator $T_\tau(.)$ by the following definition

$$T_\tau(y)(n):=\sum_{i=0}^{\infty}\sum_{j=0}^{n-1}y(n-j+i)f(j)g(i)\tau^{-j+i}+\tau^{-n}\sum_{j=n}^{\infty}f(j)\,.$$

$T_\tau(.)$ is an operator mapping from set of the bounded sequence to the set of bounded sequences. Furthermore

$$|T_\tau(y_1)(n)-T_\tau(y_2)(n)|\le\sum_{i=0}^{\infty}\sum_{j=0}^{n-1}|y_1(n-j+i)-y_2(n-j+i)|f(j)g(i)\tau^{-j+i}\,,$$

hence

$$\|T_\tau(y_1)-T_\tau(y_2)\|_\infty\le\|y_1-y_2\|_\infty\cdot\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}f(j)g(i)\tau^{-j+i}=\|y_1-y_2\|_\infty\cdot H(\tau)$$

with notation $\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} f(j)g(i)\tau^{-j+i} = H(\tau)$. We note that $H(1)=1$.We prove that in case $\mu_g > \mu_f \quad H(\tau)<1$ for appropriate values of $\tau<1$. As $H(\tau)=\sum_{j=0}^{\infty} f(j)\tau^{-j}\sum_{i=0}^{\infty} g(i)\tau^{i}$,

$$H'(\tau)=\sum_{j=1}^{\infty} f(j)(-j)\tau^{-j-1}\sum_{i=0}^{\infty} g(i)\tau^{i}+\sum_{j=0}^{\infty} f(j)\tau^{-j}\sum_{i=1}^{\infty} g(i)i\tau^{i-1},$$

$H'(1)=-\sum_{j=0}^{\infty} j\cdot f(j)+\sum_{i=0}^{\infty} i\cdot g(i)=-\mu_f+\mu_g>0$, there exists a $0<\tau^*<1$ for which $H(\tau)<1$, $\tau^*\le\tau<1$. Choose $\tau_0=\max(\tau_1,\tau^*)$. As $H(\tau_0)<1$, hence $T_{\tau_0}(.)$ is a contraction and the set of bounded sequences is complete, then there is a unique solution of Eq.(4.1) in the set of bounded sequences. As a consequence, there is a unique solution of Eq.(3.1) with property $|x(n)|\le{\tau_0}^n K$. It means that there is a unique exponentially bounded solution to (3.1).

The solution of (3.1) in the set of bounded sequences is not unique, as $x(n)\equiv 1$ is a solution and according to the previous statement under some condition we have an exponentially bounded solution as well. We will see explicit examples as well in among the special cases. But if $\mu_g>\mu_f$, then applying technique presented in [5] one can prove that the sequence defined in (2.2) have the following limit property: $\lim_{n\to\infty} x(n)=0$.

We have the conjecture, but unfortunately we could not prove up to now, that the solution of (3.1) is unique in the set of sequences tending to zero. If this conjecture is true, then Theorem 4.1 states that $x(n)$ defined in (2.2) tends to zero with exponential rate under the assumption of Theorem 4.1. This is underlined by the explicit solutions in special cases as well.

5 Explicit solutions for special cases

In this section we give analytical solutions for Equation (2.2) in the special case when the time intervals between the consecutive fillings have geometric distribution.

In insurance mathematics in the continuous case, exponential distribution for the consecutive filling times is very important and often investigated, this model is called compound Poisson risk process [9]. As the discrete analogue of the exponential distribution is the discrete geometrical distribution, we solve the difference equation in this special case.

Theorem 5.1

Consider the special case $f(j)=(1-\bar{f})\bar{f}^{\,j}, \quad j=0,1,2,...$, supposing that $0<f(0)=1-\bar{f}<1$. If $\mu_f<\mu_g$, then there exists a solution of Eq.(3.1) of the form $x(n)=\mu^n$, where $0<\mu<1$. If $\mu_f\ge\mu_g$, then there is no solution of Eq.(3.1) of form $x(n)=\mu^n$, where $0<\mu<1$.

Proof:

Let us try to find a solution of Eq.(3.1) in the form $x(n)=\mu^n$. Substituting it into Eq.(3.1) we get

$$\mu^n=\sum_{i=0}^{\infty}\sum_{j=0}^{n-1}\mu^{n+i-j}(1-\bar{f})\bar{f}^{\,j}g(i)+\sum_{j=n}^{\infty}(1-\bar{f})\bar{f}^{\,j}\ . \tag{5.1}$$

By elementary computations one can see that $\bar{f}\neq\mu$. Using this fact when summarizing, after summing up and arranging Eq.(5.1) we get

$$\left(\mu^n-\bar{f}^{\,n}\right)\left(\mu-\bar{f}-\sum_{i=0}^{\infty}\mu^{i+1}g(i)(1-\bar{f})\right)=0 \tag{5.2}$$

As Eq.(5.2) has to be hold for any values $n\in N$, hence the following characteristic equation has to be satisfied:

$$k(\mu)=\mu-\bar{f}-\sum_{i=0}^{\infty}\mu^{i+1}g(i)(1-\bar{f})=0\,. \tag{5.3}$$

We note that this equation is the discrete analogue of the equation (14) presented in [7].

It can be easily proved that $k(0)<0$, $k(1)=0$, $k'(\mu)=1-\sum_{i=0}^{\infty}(i+1)\mu^{i}g(i)(1-\bar{f})$, and $k'(\mu)$ is a monotone decreasing function of μ. Furthermore $k_1'(1)=\bar{f}-(1-\bar{f})\sum_{i=0}^{\infty}ig(i)$. As $E(Y_k)=\sum_{i=0}^{\infty}ig(i)$ is supposed to be finite and $E(t_k)=\sum_{j=0}^{\infty}jf(j)=\frac{1}{1-\bar{f}}-1=\frac{\bar{f}}{1-\bar{f}}<\infty$, we can consider two different cases.

I. $E(t_k)=\mu_f<E(Y_k)=\mu_g$ that is $k'(1)<0$. Now there exists a $0<\mu<1$ for which $k(\mu)=0$. $E(t_k)<E(Y_k)$ expresses that the expectation of the amount of material filled into the storage during unit time interval is more than the material withdrawn from the storage during unit of time.

II. $E(t_k)=\mu_f\geq E(Y_k)=\mu_g$ that is $k'(1)\geq 0$. In this case, if $k(\mu)=0$ for any $0<\mu<1$, then $k(\nu)\equiv 0$ for $\mu\leq\nu\leq 1$ which is a contradiction. Hence in this case there is no solution of Eq.(5.3) in (0,1).

We note if $\mu_f<\mu_g$ with $f(j)=(1-\bar{f})\bar{f}^{\,j}$, $j=0,1,2,\ldots$ $0<f(0)=1-\bar{f}<1$ conditions of Theorem 4.1 hold hence unique exponentially bounded solution exists.

Special case 1. Let us consider the case $f(j)=(1-\bar{f})\bar{f}^{\,j}$, $j=0,1,2,\ldots$, assuming that $0<f(0)=1-\bar{f}<1$. If $g(i)=\begin{cases}1 & if \quad i=1\\ 0 & if \quad i\neq 1\end{cases}$, what is valid in the case of constant $Y_k=1$, then the characteristic equation (5.3) is $k(\mu)=\mu-\bar{f}-\mu^2(1-\bar{f})=0$. Now $\mu_1=1$, $\mu_2=\frac{\bar{f}}{1-\bar{f}}$. If $\mu_f<\mu_g=1$, then $\mu_2<1$, in the opposite case $\mu_2\geq 1$.

Special case 2. Let us consider the case $f(j)=(1-\bar{f})\bar{f}^{\,j}$, $j=0,1,23\ldots$with $0<f(0)=1-\bar{f}<1$, and let $g(i)=(1-\bar{g})\bar{g}^{\,i}$, $i\geq 0$, with $0<\bar{g}<1$. Then the characteristic equation is expressed as $k(\mu)=\mu-\bar{f}-\mu\sum_{i=0}^{\infty}(\mu\bar{g})^{i}(1-\bar{g})(1-\bar{f})$. As the case $\mu\bar{g}=1$ leads us to a contradiction, we can make computations only for $\mu\,\bar{g}\neq 1$. Arranging the characteristic equation we get $-\bar{g}\mu^2+(1+\bar{f}\bar{g}-(1-\bar{g})(1-\bar{f}))\mu-\bar{f}=0$. This equation has again two positive solutions, namely $\mu_1=1$ and $\mu_2=\frac{\bar{f}+\bar{g}-|\bar{f}-\bar{g}|}{2\bar{g}}$. $\mu_2<1$ is satisfied if and only if $\mu_f<\mu_g$, that is $\bar{f}<\bar{g}$.

We note that one can prove that each of the bounded solutions of special cases 1 and 2 has a form $x(n)=c_1+c_2\mu^n$ where $c_1+c_2=1$. If the limit of the solution has to be zero, then we will have a unique bounded solution for Eq.(3.1) in these special cases assuming $\mu_f<\mu_g$. This fact coincides with our conjecture given in the previous section, namely the solution is unique in the set of sequences tending to zero.

We can summarize our results in the followings:

Theorem 5.2

Let $f(j)=(1-\bar{f})\bar{f}^{\,j}$, $j=0,1,2,\ldots$ $0<f(0)=1-\bar{f}<1$, and $g(i)=\begin{cases}1 & if \quad i=1\\ 0 & if \quad i\neq 1\end{cases}$ and assume $\mu_f<1$, that is $\bar{f}<0.5$. Now $x(n)$ defined by (2.2) can be expressed as

$$x(n) = \left(\frac{\overline{f}}{1-\overline{f}} \right)^n, \; n = 0,1,2,... \tag{5.4}$$

Let $f(j) = (1-\bar{f})\bar{f}^j$, $j = 0,1,2,...$ $0 < f(0) = 1-\bar{f} < 1$, and $g(i) = (1-\bar{g})\bar{g}^i$, $i \geq 0$, with $0 < \bar{g} < 1$ and assume $\overline{f} < \overline{g}$. Now $x(n)$ defined by (2.2) can be expressed as

$$x(n) = \left(\frac{\overline{f}}{\overline{g}} \right)^n \; n = 0,1,2,... \; . \tag{5.5}$$

6 Computational examples

Finally we show some computational examples which illustrate our results.

First we choose $f(j) = (1-\bar{f})\bar{f}^j$, $j = 0,1,2,...$ $\overline{f} = 1/1.15$, $g(i) = (1-\bar{g})\bar{g}^i$, $i \geq 0$, with $0 < \bar{g} = 10/11$. These parameters corresponds to the parameters of the process presented in [6] on Figure 3.

Now $\overline{f} < \overline{g}$. One can see the probabilities of running out of material applying (5.5), and the probabilities presented in [6] on Figure 3, namely the function $\psi_2(x) = 1 - R_2(x) = e^{-(\frac{1}{\mu_f c} - \mu_g)x}$ according to the formula (15) in [6]. One can state that there are small differences between the numerical results.

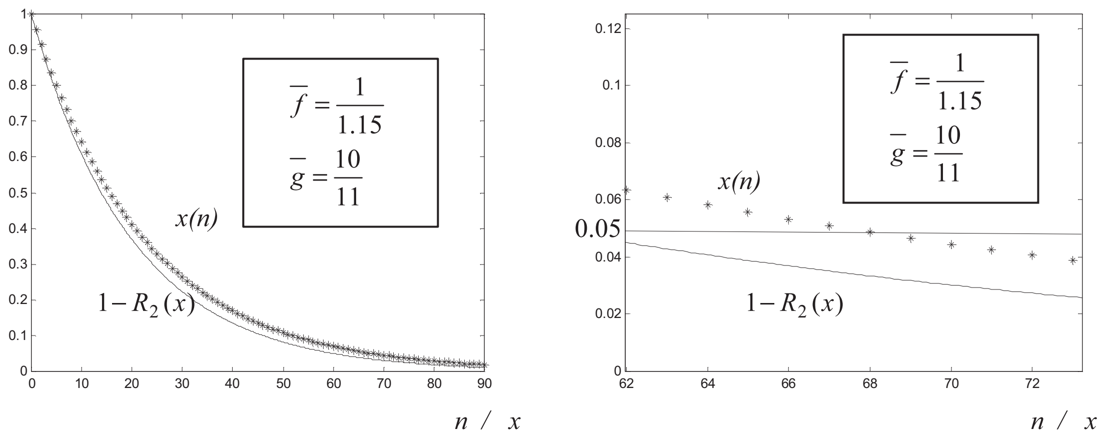

Figure 6.1. Analytical solution of Eq. (3.1) $x(n)$ (*) in case of discrete geometrical distribution of t_k, Y_k and the underflow probabilities $1 - R_2(x)$ (__) in case of exponential distribution of t_k, Y_k

We use this example to determine the initial amount of material appropriate to the reliability level 0.95.

If we want to know the smallest *n* for which the probability of having enough material is at least 0.95, we have to find the smallest n for which $1 - x(n) \geq 0.95$. If you use geometrical distribution for the distribution of consecutive filling times and the amount of material with the above given parameters, we get $x(n) = \left(\frac{\overline{f}}{\overline{g}} \right)^n = (11/11.5)^n$ $n = 0,1,2..$, hence $n = 68$, as it can be seen on the Figure 6.1.

Let our second example be the following: $f(j) = (1-\bar{f})\bar{f}^j$, $j = 0,1,2,...$ with $\overline{f} = 0.8$, and let the distribution of Y_k binomial distribution with parameters $m = 10$, $p = 0.5$, $P(Y_k = i) = g(i) = \binom{10}{i}\left(\frac{1}{2}\right)^{10}$ $i = 0,1,2,...,10$.

In this case we do not have explicit formula for $x(n)$. But applying Theorem 5.1. we know that there exists solution in form $x(n)=\mu^n$, and μ is the solution of Eq. (5.1). This equation can not be solved analytically in this case, hence we solved it numerically. We get $\mu=0.9422$, and the solution $x(n)$ can be seen on Figure 6.2.

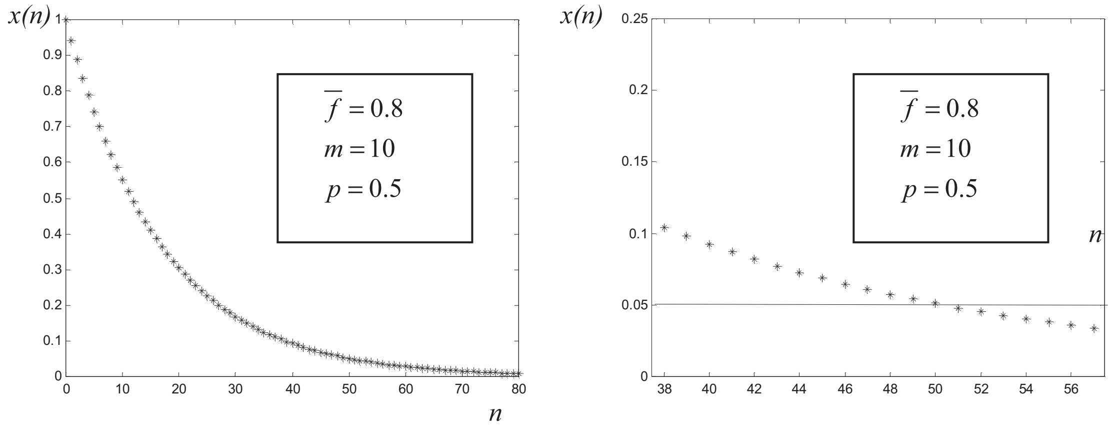

Figure 6.2. Solution of Equation (3.1) $x(n)$ in case of geometrical distribution of t_k and binomial distribution of Y_k .

If we would like to determine the initial amount of material appropriate to the reliability level 0.95, we have to determine the smallest n for which $1-x(n)\geq 0.95$ and we get $n=51$.

7 Summary

A discrete mathematical model is presented and analyzed for sizing intermediate storages to a given reliability level. First we introduced the probabilities of running out of material in the function of initial amount of material as a sequence. We set up and proved the difference equations satisfying by the probabilities. We proved the existence and the uniqueness of the solution in the set of exponentially bounded solutions under some conditions and we presented an example when the bounded solution is not unique. We solved the equation analytically in special cases and used the analytical solution for solving the original sizing problem. We compared the analytical solution of the discrete model to the analytical solution of the appropriate model with continuous distributions and the coincidence of the results enforces that the discrete model can be applied as an approximation of the more complicated continuous model.

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