Optimal Sensor placement in flexible structures: Energy vs. Information Based criteria

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Abstract. This paper is concerned with the optimal placement of sensors (OSP) on flexible dynamical structures. In order to get as much information as possible about the system state at any given time - also in terms of robustness with respect to numerical issues and modeling errors or sensor noise - the optimal placement of the sensors is an important system design task. Two different approaches for OSP are considered: The first one maximizes the output information using an iterative elimination algorithm, while the second one evaluates the signal output energy that can be received by the sensors. The contribution of this paper is to emphasize the basic mathematical consistency of these two approaches as well as their application on a simple theoretical example.

1 Introduction

The static and dynamic behavior of a physical system is described by system variables, which allow for a description of the system state at any given time. However, the system states are usually not directly available and appropriate variables have to be measured by physical sensors to guarantee observability of all system states. While classical observability criteria only give qualitative information on observability (yes or no), more advanced methods yield quantitative results in terms of robustness with respect to numerical issues and modeling errors or sensor noise. Thus, both the careful choice of the sensing principle as well as optimal placement of the sensors are important system design tasks.

This paper is concerned with the optimal placement of sensors (OSP) on flexible dynamical structures approximated by a finite- dimensional representation, a frequently discussed research topic during the last ten years [1] and earlier [2]. The goal is to establish a criterion which specifies where to place displacement, force, inertial acceleration, or other sensors, such that either the energy or the information is maximized in the sensor outputs. Typical applications cover modeling, identification, fault detection, and especially active control of large flexible structures such as bridges [7] [8], rail wagons [9], or space structures [10]. Therefore, various methods and criteria for OSP and optimal actuator placement, which is basically a dual task, have been developed.

Some authors use the Hankel singular values (HSV) of the structure to determine the degree of controllability and observability for a multiple set of actuators and sensors [3],[4]. In [5], the minimum eigenvalues of the controllability and observability gramian are maximized to minimize the input energy to reach a given state or to maximize the output energy generated by a given state, respectively. In [6], another methodology based on quantitative measures for observability and controllability is proposed, which is also suitable to evaluate the possibility of typically occurring component failures. The H_2, H_∞ or Hankel modal norm as a measure for the system response intensity on standard excitations like white noise or unit impulse is used in [11] to define a proper placement index.

In this work two state of the art sensor placement criteria are considered, which are based on different approaches. The first method, treated in section 2, is information based and analyzes the information content in sensor measurements via the output mode shape matrix. Section 3 contains the second method for OSP, which is energy based. Here, positions on the structure are determined where the output signal to noise ratio can be maximized [13],[14]. The basic concepts of both methods are compared in section 4, showing the essential consistence of the two approaches in mathematical form and also by an illustrating example. In section 5 conclusions and prospects on future work are given.

2 Information based approach for OSP

Optimal sensor placement techniques are extensively discussed in the contributions [7] [8]. A short overview of the effective independence (EFI) method follows in this section 2, adopted from [7] [8].

The aim of the EFI method is to select measurement positions that make the mode shapes of interest as linearly independent as possible while containing sufficient information about the target modal responses in the measurements. The method originates from estimation theory. The use of a sensitivity analysis of the parameters to be estimated leads to the maximization of the Fisher information matrix, measured by the determinant or the trace, which in fact is an equivalent minimization of the condition number of the information matrix. Technically, it is reflected in a coefficients' covariance matrix (the covariance matrix of the estimate error of the modal coordinates is minimized). The number of sensors is reduced from an initially large candidate set in an iterative manner by removing sensors from those places which contribute least among all the candidate sensors to the linear independence of the target modes. In the end, the required candidate sensors are preserved as the optimal sensor set. As a useful guideline toward the selection of a suitable number of sensors, the determinant of the Fisher information matrix can be plotted with respect to the number of sensors. If a considerable drop is identified, further reduction

The sensor placement problem can be investigated from the uncoupled modal coordinates of the governing structural equations as follows:

$$\ddot{q}_i + M_i^{-1} \cdot D_i \cdot \dot{q}_i + M_i^{-1} \cdot K_i \cdot q_i = M_i^{-1} \cdot \Phi^T \cdot B_0 \cdot u \tag{1}$$

$$y = C_q \cdot \Phi \cdot q + C_v \cdot \Phi \cdot \dot{q} + \varepsilon, \tag{2}$$

where q_i is the i^{th} modal coordinate and also the i^{th} element of the vector q in equation (2), M_i, K_i and D_i are the corresponding i^{th} modal mass, stiffness and damping matrices, respectively, $C_q \cdot \Phi$ is the mode shape matrix with its i^{th} column as the i^{th} mass-normalized mode shape, B_0 is a location matrix formed by ones (corresponding to actuators) and zeros (no loadings), specifying the positions of the force vector u and C_q, C_v are the location matrices for the displacement and velocity outputs, respectively. y is the output column vector indicating which positions of the structure are measured, and the column vector ε adds Gaussian white noise with zero mean and a variance of σ^2.

From the output measurement the EFI analyzes the covariance matrix of the estimation error for an efficient unbiased estimator as follows:

$$E\left[(q-\hat{q}) \cdot (q-\hat{q})^T\right] = \left[\left(\frac{\partial y}{\partial q}\right)^T \cdot \left[\sigma^2\right]^{-1} \cdot \left(\frac{\partial y}{\partial q}\right)\right]^{-1} = Q^{-1}. \tag{3}$$

In (3), Q is the Fisher information matrix, E denotes the mean values, and $\hat{q}$ is the efficient unbiased estimator of q. Maximizing Q will result in the best state estimate of q. The EFI coefficients of the candidate sensors are computed by the following formula:

$$E_D = [\Phi \cdot \Psi] \otimes [\Phi \cdot \Psi] \cdot \lambda^{-1} \cdot 1, \tag{4}$$

where $\otimes$ represents a term-by-term matrix multiplication, 1 is an $n \times 1$ column vector with all elements equal to 1. Ψ denotes the eigenvector matrix according to the eigenvalues on the diagonal of the λ matrix. E_D are the EFI indices, which evaluate the contribution of a candidate sensor location to the linear independence of the modal partitions. The selection procedure is to sort the elements of the E_D coefficients, and to remove the smallest one at a time. The E_D coefficients are then updated according to the new modal shape matrix, and the process is repeated iteratively until the number of remaining sensors equals a preset value. The remaining DOFs serve as the measurement locations.

The main technical constraint for the EFI method is that the number of finally retained sensors must be greater or equal to the number of modes selected. For instance, for the first three modes being of interest, one can receive optimal 20, 5 or 3 locations, but no meaningful results can be achieved for 2 sensors. This limitation is due to the evaluation of the E_D vector coefficients, based on the eigenvalues and eigenvectors of the Fisher information matrix Q (singular values of target mode shape matrix). If the number of considered measurements is lower than the number of modes, the Fisher information matrix becomes singular (some singular values become very close to zero).

3 Energy based approach for OSP

The second approach for optimal sensor placement considered in this paper is based on maximizing the amount of the energy of the signal that can be received by the sensors [13],[14]. In order to find optimal sensor positions, a state space system representation in modal coordinates has some favorable advantages, especially in combination with flexible structures. Therefore, the system given in (1),(2) can easily be transformed as outlined in [11]. An appropriate state vector for the first n modes is given by

$$x = [\omega_1 q_1, \dot{q}_1, \omega_2 q_2, \dot{q}_2, \ldots, \omega_n q_n, \dot{q}_n]^T, \tag{5}$$

with the corresponding state-space representation

$$\dot{x} = Ax + Bu \tag{6}$$

$$y = Cx, \tag{7}$$

where u is the $r \times 1$ input vector and y the $s \times 1$ output vector (r as the number of inputs and s as the number of outputs). For the $2n \times 2n$ modal system matrix A, $2n \times r$ modal input matrix B and the $s \times 2n$ modal output matrix C holds

$$A = diag\,(A_i)\,, \quad A_i = \begin{bmatrix} 0 & \omega_i \\ -\omega_i & -2\zeta_i\omega_i, \end{bmatrix} \tag{8}$$

$$B = [0, b_1, 0, b_2, \ldots, 0, b_i, \ldots, 0, b_n]^T, \tag{9}$$

$$C = \left[\frac{c_{q1}}{\omega_1}, c_{v1}, \frac{c_{q2}}{\omega_2}, c_{v2}, \ldots, \frac{c_{qi}}{\omega_i}, c_{vi}, \ldots, \frac{c_{qn}}{\omega_n}, c_{vn}\right] \tag{10}$$

where ω_i and ζ_i denote the natural frequency and the modal damping of the i^{th} mode. The block diagonal form of the system matrix is caused by the special choice of the state vector.

The energy possible to receive at the respective sensor position is evaluated for two different cases of system inputs: transient and persistent excitation. In both cases, the aim is to maximize the signal to noise ratio. For transient excitation, a maximum energy problem can be defined

$$J = \int_0^\infty y^T(t)y(t)dt \quad \rightarrow \quad \max, \tag{11}$$

in order to make the signal energy for a state transformation as large as possible. If the system is released from an initial state $x(0) = x_0$ the maximum energy is given by

$$J_{max} = x_o^T W_o x_0, \tag{12}$$

where

$$W_o = \int_0^\infty e^{A^T t} C^T C e^{At} dt \tag{13}$$

is the observability gramian as a function of time. For asymptotically stable systems a unique positive semi-definite matrix exists, which satisfies the Lyapunov equation

$$A^T W_o + W_o A + C^T C = 0. \tag{14}$$

It is important to note, that W_o depends on the particular choice of state variables. Considering flexible structures with well separated natural frequencies and small structural damping $\zeta << 1$ and using the modal state space representation as defined in (5)-(10), a diagonally dominant observability gramian

$$W_o \cong diag\,(w_{ci} I_2) \tag{15}$$

results, where I_2 is the 2×2 identity matrix and w_{oi} the modal observability factor. The latter can be expressed in closed form

$$w_{oi} = \frac{\|C_i\|_2^2}{4\zeta_i\omega_i}, \tag{16}$$

where $\|C_i\|_2$ indicates the H_2 norm of the $s \times 2$ block of the output matrix for the i^{th} mode. The off-diagonal elements in (15) are comparatively small and therefore negligible. Thus, for a transient system excitation, the task is to descry those sensor positions where the diagonal elements of the observability gramian have their maximum.

As a second approach, a continuing system perturbation is considered. Under the assumption of white noise excitation, the mean square value of the system output is given by

$$E[y^T(t)y(t)] = tr[(C^T C)X(t)] \tag{17}$$

with $X(t)$ as the time-varying covariance matrix of the state vector $x(t)$. For steady state, a time invariant solution $X(t) = const.$ exists, which satisfies the Lyapunov equation

$$AX + XA^T + bb^T = 0, \tag{18}$$

where the $2n \times 1$ vector b is given by:

$$b = [1\ 0\ 1\ 0\ \ldots\ 1\ 0]^T. \tag{19}$$

For flexible structures and the given state vector, X is again diagonally dominant

$$X \cong diag\left(\frac{1}{4\zeta_i\omega_i} I_2\right). \tag{20}$$

Therefore, for the mean square value of the system output holds

$$E[y^T(t)y(t)] = \sum_{i=1}^{n} \frac{\|C_i\|_2^2}{4\zeta_i\omega_i}, \tag{21}$$

which basically coincides with the result for a transient disturbance.

Using the results above, an appropriate performance index (PI) for each possible sensor location j out of a set of location candidates can be defined as follows [13]:

$$PI_j = \left(\sum_{i=1}^{2n} \lambda_{Woi}\right) \sqrt[2n]{\prod_{i=1}^{2n} (\lambda_{Woi})}. \tag{22}$$

Here, λ_{Woi} is the i^{th} eigenvalue of the observability gramian and n is the number of structural vibration modes. The first term of PI represents the total energy received by the sensors from the structure, which is essentially dominated by a few low frequency modes. The second term, the geometric mean of the eigenvalues, accounts also for higher frequency modes, since it tends to be zero even if just one mode is marginally observable. The PI is evaluated for each possible sensor location, whereas those positions with the maximum values for the PI are most suitable.

4 Information vs. Energy based OSP

In the previous two chapters, the fundamental theory of two state of the art sensor placement criteria, which use different approaches in order to find optimal sensor positions on flexible dynamical structures, was described. Note that for both approaches formulation of a sensor principle is sufficient to obtain the optimal sensor positions. However, existing additional system knowledge could be introduced by appropriate input/output weighting. This chapter is dedicated to a comparison of the basic results given above, both mathematically as well as on a simple example.

Concerning the information based approach, the information content for the considered set of sensors is measured by either the trace or the determinant of an underlying Fisher information matrix Q (3). The contribution of each single sensor location k is expressed through the E_D vector:

$$E_D(k) = \sum_{i=1}^{n} \frac{{\phi_{ik}}^2}{\lambda_{Qi}} \tag{23}$$

where λ_{Qi} is the i^{th} eigenvalue of the Fisher information matrix. For structures with small damping and well separated natural frequencies, the Fisher information matrix is diagonally dominant and for the eigenvalues holds:

$$\lambda_{Qi} \cong \sum_{k=1}^{s} {\phi_{ik}}^2 \tag{24}$$

On the contrary, the signal energy at a single sensor position can be determined through the observability gramian as a quantitative measure. Twice the total energy E_{tot} received by the sensor k is given by the first term in (22)

$$2E_{tot}(k) = \sum_{i=1}^{2n} \lambda_{Woik}, \tag{25}$$

where λ_{Woik} is given by

$$\lambda_{Woik} \cong \frac{{\phi_{ik}}^2}{4\zeta_i\omega_i}. \tag{26}$$

Comparing (23) and (25) it turns out that E_D and E_{tot} coincide up to a scalar factor. Therefore, the only difference between the two methods for OSP is given by the product term in (22), which represents a simple extension of the total energy term in order to avoid loss of observability of higher frequency modes.

As an illustration of the given comparison, a clamped beam is considered as defined in [11] p.3, consisting of 15 elements (14 nodes). For the sake of simplicity, only the first four modes are modeled. In Fig. 1 the associated target mode shapes are depicted.

Fig. 2 presents the optimal sensor positions based on the effective independence method E_D and the total signal energy E_{tot} (in right part of figure) respectively. Result of EFI method (plotted in left part of Fig. 2) is the optimal set of sensors, there is no qualitative measure of sensor information/energy content as in Energy approach (plotted in right part of Fig. 2). Both methods identify the position in the middle of the beam as the optimal one for sensor

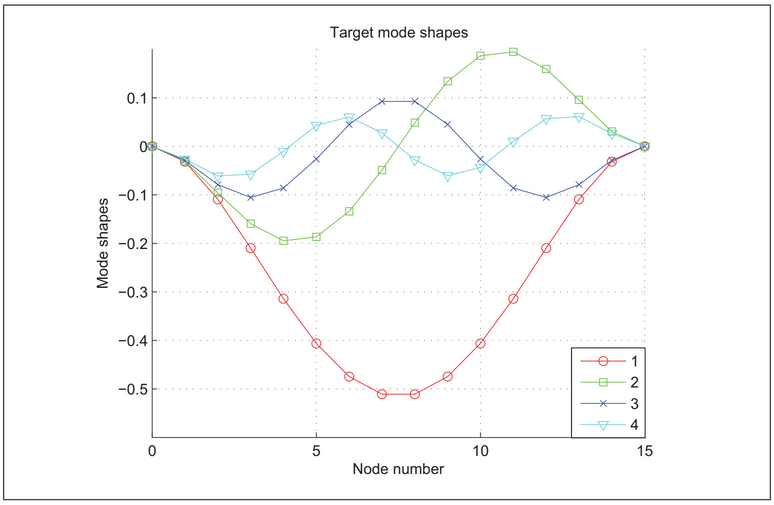

Figure 1: Mode shapes of first 4 modes of clamped beam.

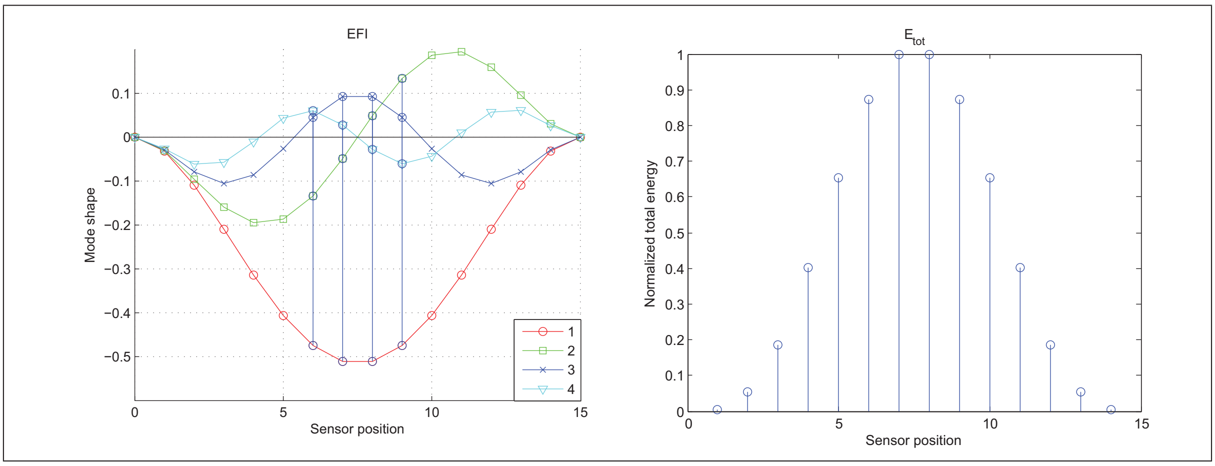

Figure 2: Optimal sensor positions determined by the effective independence method / the total signal energy component

placement. This is due to the high influence of the first order mode. Both approaches do not account for the fact, that the 2-nd and 4-th mode are hardly observable at that position.

However, the PI as it is defined in (22) does also account for the higher frequency modes. Positions where at least one mode is marginally observable are penalized by the second term of the PI. Considering the OSP results determined by the PI (Fig. 3), the optimal positions are located symmetrically at points 6 and 9.

5 Conclusions and Prospects

In this paper, a comparison of two different approaches for optimal sensor placement on flexible mechanical structures is given. On the one hand an information based method is considered, which analyzes the information content in the sensor measurements by use of the output mode shape matrix. The second approach is based on the determination of the signal output energy in order to maximize the signal to noise ratio. Therefore, the observability gramian serves as a quantitative measure for evaluation of the signal energy. A direct comparison of both approaches shows the basic coincidence of them. Up to a scalar factor, the vector expressing the information content and the vector containing the total energy are equal, which is also verified on a simple example. Furthermore, it is shown that an appropriate definition of a performance index is useful with respect to the observability of higher frequency modes, since considering only the signal information content or total energy, respectively, can lead to sensor positions unable to observe certain higher order modes.

Basically, this paper focuses on the fundamental principles of the two methods and their comparison. However, several extensions of these methods are possible in order to meet some practical requirements. For example, the performance index for optimal sensor placement presented in this paper depends on the appropriate choice of state variables, which makes its use difficult for complex structures in arbitrary system coordinates. In order to avoid these difficulties, a possible modification of the performance index will be presented in a future work, which additionally uses the H_2 norm to evaluate the response intensity of the system on a standard excitation. A placement criterion will be defined suitable for both sensor as well as actuator placement on flexible structures. In the EFI

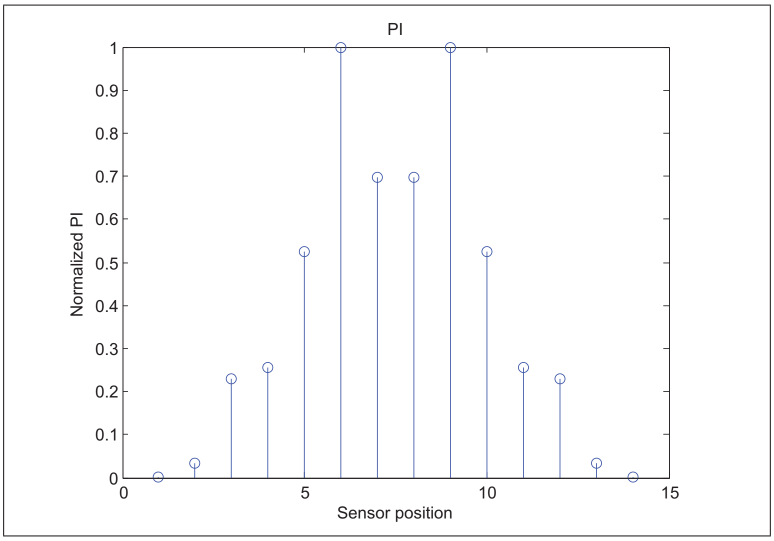

Figure 3: Optimal sensor positions determined by the PI

algorithm the sensor selection is based on a preselection of the most wanted structural modes. But it can also be of interest to suppress the observation of unwanted higher order modes, while simultaneously the observability of low frequency modes should be as high as possible (modal sensor idea). For this purpose a modified criterion will be presented in a future work.

6 References

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