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MINIMAL PARAMETERIZATIONS IN DIAGNOSIS–ORIENTED MULTIVARIABLE IDENTIFICATION

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Abstract. Dynamical systems can be described by several classes of models and it is also possible to define, inside most of these classes, equivalence relations that have no influence on the input–output behaviour of the system. For most applications it is thus possible to use any model inside an equivalence class and, in fact, this possibility is widely used in the solution of analysis and synthesis problems (e.g. changes of basis in the state space for state space models).

When the models are used for identification purposes, however, the use of non minimally parameterized models leads to a larger dispersion of the parameter estimates with obvious negative effects on fault detection and localization procedures based on the values of the parameters of identified models.

This paper defines the minimal number of parameters that can describe a dynamical system on the basis of the invariance properties of equivalence relations defined inside the considered class of models and shows, by means of Monte Carlo simulations, that also modest overparameterizations can lead to remarkable increases in the dispersion of the parameter estimates.

1 Introduction

A dynamical system is a mathematical entity that can be described, completely or in part, by means of several families of models like state–space ones, transfer functions, polynomial models etc. Inside most families of models it is possible to consider equivalence relations that define partitions into equivalence classes; every specific system can thus be described by any of the infinite models belonging to the associated class. The selection of specific models inside a class to solve analysis or synthesis problems is an established praxis in System Theory because it can lead to a reduction of the complexity of the involved procedures and/or to a better numerical robustness.

In applications like simulation, filtering or control, any model inside an equivalence class can be used without drawbacks. It must however be considered that only some of the models inside a specific class are parameterized by the minimal number of scalars necessary to describe the associated system so that, in all models with a non minimal parameterization, a variation in one parameter could be compensated by variations in other parameters without exiting from the class i.e. without introducing any variation in the associated system. This aspect is very important when the models are deduced from the system behaviour (identification) and used for fault diagnosis by adopting fault detecting criteria based on the parameter values. In these applications it is important to make reference to models described by the minimal number of parameters in order that their variations can be interpreted only as variations of the underlying system.

The identification of dynamic processes can be based not only on a variety of models, like in realization, but also on a variety of assumptions on the stochastic context of the data generation process. Even when the class of models to be used and the assumptions on the data have been defined, the purpose of the identification experiment can suggest the use of specific cost functions leading to deduce different models from the same set of data.

The prevailing use of identified models concerns prediction and control so that equation–error models and quadratic functions of the prediction error remain important and widely used tools. Identified models play, however, also other roles of increasing importance; they are used, for instance, in the solution of fault localization and diagnosis problems. These applications, in turn, can rely on different properties of the models, like eigenvalue or pole patterns, associated congruence relations or excursion of the parameter values. This last approach can be particularly significant when the parameter variations can be associated to physical properties of the process and poses some non trivial challenges when the process to be identified is a multivariable one and a non minimal parameterization is selected. In fact, if a SISO model given by a transfer function

$$G(z) = \frac{p(z)}{q(z)} \tag{1}$$

or by a polynomial difference model

$$q(z)\,y(t) = p(z)\,u(t), \tag{2}$$

where $p(z)$ and $q(z)$ are polynomials in the unitary advance operator z and $u(t)$ and $y(t)$ denote the model input and output is considered, when the order is correct and no common factors are present in $p(z)$ and $q(z)$, the model parameterization is necessarily minimal. Similar considerations can be repeated for MISO systems where the polynomial model (2) assumes the form

$$q(z)\,y(t) = \sum_{i=1}^{r} p_i(z)\,u_i(t), \tag{3}$$

where r denotes the number of inputs. In the multivariable case, model (2) assumes the form

$$Q(z)\,y(t) = P(z)\,u(t), \tag{4}$$

where, by denoting with m the number of outputs, $Q(z)$ and $P(z)$ are $(m \times m)$ and $(m \times r)$ polynomial matrices endowed with a structure that cannot be described simply by the model order. Moreover, in this case, the minimality of the parameterization is not assured by the absence of common left factors between $Q(z)$ and $P(z)$. Some approaches, like the subspace one, lead to state space models

$$x(t+1) = A\,x(t) + B\,u(t) \tag{5}$$

$$y(t) = C\,x(t) \tag{6}$$

but do not offer any choice on the state space basis so that the number of parameters appearing in the matrices (A, B, C) is not minimal.

In the context of fault diagnosis for multivariable systems it is thus relavant to analyse the conditions for the minimality of the parameterization of the models used in their identification. It is also important to evaluate the level of uncertainty that can be expected when non minimally parameterized models are used.

The purpose of this paper is to describe some properties of invariant functions for equivalence relations and to show, by means of Monte Carlo simulations, that the use of non minimally parameterized models leads, in identification, to a dispersion of the estimated parameters that makes them unsuitable for fault detection and localization [2].

More precisely, Section 2 describes the properties of complete sets of invariants for equivalence relations and the bijection between the image of an equivalence class in a complete set of independent invariants and a set of canonical forms. Section 3 applies these results to polynomial multivariable models and describes a set of minimally parameterized multivariable canonical descriptions. Section 4 shows, by means of Monte Carlo simulations, that the parameter dispersion of these models is remarkably lower than the dispersion of the parameters of other models belonging to the same equivalence class but described by a non minimal number of parameters. Short final remarks are finally given in Section 5.

2 Complete sets of independent invariants for equivalence relations

Definition 1 – Denote a set with X and an equivalence relation defined on X with E. Then denote with S a second set and with $f : X \to S$ a function. If x' and x'' are two elements of X, and f is such that $x' \, E \, x''$ implies $f(x') = f(x'')$ then f is called an *invariant* for E. Moreover if $f(x') = f(x'')$ implies $x' \, E \, x''$ then f is called a *complete invariant* for E.

If f is a complete invariant for E then all the elements of X belonging to the same equivalence class have the same image in f; moreover, these classes coincide exactly with the inverse images in f of the elements of the image (or range) of f. There exists, therefore, a bijection between the quotient set X/E and the image of a complete invariant for E.

Definition 2 – A set of invariants $f_1, \ldots, f_n$, $f_i : X \to S_i$ for E is called a *complete set of invariants* for E if the function $f = (f_1, \ldots, f_n) : X \to S_1 \times \ldots \times S_n$ defined by $x \to (f_1(x), \ldots, f_n(x))$ is a complete invariant for E.

Definition 3 – A set of invariants for E, $f_1, \ldots, f_n$, $f_i : X \to S_i$ will be called *independent* if the associated invariant $f = (f_1, \ldots, f_n) : X \to S_1 \times \ldots \times S_n$ is surjective.

This condition, which is more restrictive than that given in [6], implies that no invariant f_i can be expressed as a function of the others. This last condition, however, is much weaker than the given definition of independence. A complete set of independent invariants for E is also called a *basis* for E on X.

Lemma 1 [6, 1] – Let $f : X \to S$ be a complete surjective invariant for E. Then every other invariant for E can be uniquely computed from f.

Proof: Let $f : X \to S$ and $g : X \to T$ be, respectively, a complete surjective invariant and a generic invariant for E. Commutativity in the diagram of Fig. 1 can be obtained if and only if for every element s of S the function h is defined as $h(s) = g(x)$ where x is any element of X such that $f(x) = s$. Since f is complete and surjective and g is an invariant, h is well defined for all the elements of S.

Corollary 1 – Let $f : X \to S$ be a complete set of independent invariants for E. Then every other invariant for E can be uniquely computed from f.

$$
\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "g"'] & & S \arrow[dl, "h"] \\
& T &
\end{tikzcd}
$$

Figure 1

Property 1 – Let $f : X \to S$ be a complete set of independent invariants for E. If $g : S \to R$ is a bijection, then $h = g \cdot f : X \to R$ is a complete set of independent invariants for E.

Definition 4 – Let E be an equivalence relation on X. A subset C of X is called a set of *canonical forms* for E if every $x \in X$ is equivalent under E to one and only one element of C; this element is *the* canonical form of x. The function $g : X \to C$ thus defined is therefore a complete invariant for E. Obviously g can be assumed surjective without loss of generality.

Let $f : X \to S$ be a complete set of independent invariants and C a set of canonical forms for E. Then (Corollary 1) there exists a unique function $h : S \to C$ such that $g = h \cdot f$. Since g is complete, h is a bijection. Moreover if $i : C \to X$ is the injection $i(c) = c$ then it follows that $h^{-1} = f \cdot i$. The following theorem has thus been proved.

Theorem 1 [6] – Let C be a set of canonical forms for an equivalence relation E on X and f a complete set of independent invariants for E. Then there exists a unique bijection between C and the image of f.

The definition of a set of canonical forms that has been considered does not imply that the elements of this set are endowed with any specific or useful property. The bijection established by Theorem 1 shows, however, that among the infinite sets of canonical forms that can be defined there exist sets whose elements are parameterized by the image of the associated equivalence class in a complete sets of independent invariants, i.e. by the minimal number of possible parameters.

3 Minimal parameterizations for dynamical systems

The properties of equivalence relations and of invariant functions for equivalence relations reported in Section 2 are quite general and do not concern, specifically, dynamic systems. They can be applied, however, to obtain minimally parameterized models for dynamical systems suitable for the identification of a process and for performing the comparison proposed in this paper.

Consider, for this purpose, the class $\mathscr{S}_o$ of polynomial input–output models (4) where no common left factors are assumed present in $Q(z)$ and $P(z)$; $\mathscr{S}_o$ can describe all completely reachable and observable systems. Consider then the equivalence relation E described by the premultiplication of $Q(z)$ and $P(z)$ by a nonsingular unimodular matrix $M(z)$.

Consider now the subset K_o, of S_o defined by the pairs $(\tilde{Q}(z), \tilde{P}(z))$ that satisfy the following conditions:

1. The polynomials on the main diagonal of $\tilde{Q}(z)$ are monic;
2. The relations between the degrees of the entries of $\tilde{Q}(z)$ are

$$
\begin{aligned}
deg\,\{\tilde{q}_{ii}(z)\} &\geq deg\,\{\tilde{q}_{ij}(z)\} \quad \text{if} \quad i > j && (7)\\
deg\,\{\tilde{q}_{ii}(z)\} &> deg\,\{\tilde{q}_{ij}(z)\} \quad \text{if} \quad i < j && (8)\\
deg\,\{\tilde{q}_{ii}(z)\} &> deg\,\{\tilde{q}_{ji}(z)\} \quad \text{if} \quad i \neq j\,; && (9)
\end{aligned}
$$

3. The relation between the degrees of the entries of $\tilde{Q}(z)$ and $\tilde{P}(z)$ is

$$
deg\,\{\tilde{q}_{ii}(z)\} > deg\,\{\tilde{p}_{ij}(z)\}\,. \tag{10}
$$

The entries of the elements of K_o will be denoted as follows:

$$
\tilde{Q}(z) = \begin{bmatrix} \tilde{q}_{11}(z) & \dots & \tilde{q}_{1m}(z) \\ \vdots & & \vdots \\ \tilde{q}_{m1}(z) & \dots & \tilde{q}_{mm}(z) \end{bmatrix} \tag{11}
$$

$$
\tilde{P}(z) = \begin{bmatrix} \tilde{p}_{11}(z) & \dots & \tilde{p}_{1r}(z) \\ \vdots & & \vdots \\ \tilde{p}_{m1}(z) & \dots & \tilde{p}_{mr}(z) \end{bmatrix} \tag{12}
$$

$$\tilde{q}_{ii}(z) = z^{\nu_i} - \alpha_{ii\nu_i} z^{(\nu_i-1)} - \ldots - \alpha_{ii2} z - \alpha_{ii1} \quad (13)$$

$$\tilde{q}_{ij}(z) = -\alpha_{ij\nu_{ij}} z^{(\nu_{ij}-1)} - \ldots - \alpha_{ij2} z - \alpha_{ij1} \quad (14)$$

$$\tilde{p}_{ij}(z) = \beta_{ij\nu_i} z^{(\nu_i-1)} + \ldots + \beta_{ij2} z + \beta_{ij1} \ . \quad (15)$$

Remark 1 – Because of relations (7)–(9) it follows that the row degrees in $\tilde{Q}(z)$ are the degrees of $\tilde{q}_{11}(z), \ldots, \tilde{q}_{mm}(z)$, i.e. $\nu_1, \ldots, \nu_m$. Moreover

$$\deg \det \tilde{Q}(z) = \sum_{i=1}^{m} \nu_i \ . \quad (16)$$

Remark 2 – The total number of significant coefficients in the entries of $\tilde{Q}(z)$ is given by

$$\ell = \sum_{i=1}^{m} \sum_{j=1}^{m} \nu_{ij} \qquad (\nu_{ii} = \nu_i) \quad (17)$$

while the total number of coefficients in the entries of $\tilde{P}(z)$ is given by

$$\sum_{i=1}^{m} \sum_{j=1}^{r} \nu_i = \sum_{j=1}^{r} n = n \times r \ . \quad (18)$$

Remark 3 – The pairs $(\tilde{Q}(z), \tilde{P}(z))$ of K_o describe multivariable systems by means of the set of m relations

$$y_i(t+\nu_i) = \sum_{j=1}^{m} \sum_{k=1}^{\nu_{ij}} \alpha_{ijk}\, y_j(t+k-1) + \sum_{j=1}^{r} \sum_{k=1}^{\nu_i} \beta_{ijk}\, \hat{u}_j(t+k-1) \ . \quad (19)$$

Theorem 2 – K_o constitutes a set of canonical forms for E on S_o.

Theorem 3 – The parameterization of the elements of K_o is the image of its equivalence classes with respect to E in a complete set of independent invariants for E. The pairs $(\tilde{Q}(z), \tilde{P}(z))$ are thus characterized by a minimal parameterization.

For a proof of Theorems 2 and 3 see [4, 5] where also the algorithms for the transformation of a generic pair $(Q(z), P(z))$ to the canonical form belonging to the same equivalence class is described. The relations between minimally parameterized state–space and input–output models can be found in [3, 4, 5].

4 Simulation results

Consider the ARX multivariable process whose deterministic part is described by the following pair

$$\tilde{Q}(z) = \begin{bmatrix} z^3 - z^2 + 0.5z - 0.2 & -0.1 \\ -0.2z - 0.2 & z - 0.6 \end{bmatrix}, \qquad \tilde{P}(z) = \begin{bmatrix} z^2 - 0.5 \\ 0.8 \end{bmatrix}. \quad (20)$$

Since the entries of $\tilde{Q}(z)$ and $\tilde{P}(z)$ satisfy conditions (7)–(10), the parameterization of this model is minimal. Model (20) consists in the following two relations that partition the whole system into two subsystems

$$(z^3 - \alpha_{113} z^2 - \alpha_{112} z - \alpha_{111})\, y_1(t) - \alpha_{121}\, y_2(t) = (\beta_{113} z^2 + \beta_{112} z + \beta_{111})\, u(t) \quad (21)$$

$$(-\alpha_{212} z - \alpha_{211})\, y_1(t) + (z - \alpha_{221})\, y_2(t) = \beta_{211}\, u(t) \, . \quad (22)$$

The input sequence used in the simulations is a pseudo random binary sequence with length $N = 100$, null mean value and variance $\sigma_u^2 = 1$; the equation error has variances $\sigma_{y_1}^2 = 0.4$ and $\sigma_{y_2}^2 = 0.35$. The first Monte Carlo simulation that has been performed consists in 100 runs where a minimally parameterized model with structure (20) has been identified by means of standard ARX algorithms (Least Squares). The dispersion of the estimates has been evaluated by considering as cost function the ratios between the standard deviations of the parameter estimates and their mean values.

The values assumed by this cost function for the parameters $\alpha_{113}, \alpha_{112}\, \alpha_{111}, \alpha_{121}, \beta_{113}, \beta_{112}, \beta_{111}$ of the first subsystem are reported in Fig. 2. The maximal value is 2.56.

Two other Monte Carlo simulations have been performed by identifying the same sequences with non minimally parameterized models belonging to the same equivalence class as model (20) but described by models whose structure can be obtained by premultiplying $\tilde{Q}(z)$ and $\tilde{P}(z)$ by the unimodular matrices

$$M_1(z) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad M_2(z) = \begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix}. \quad (23)$$

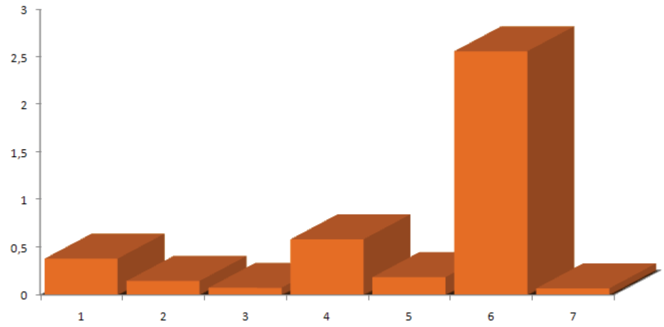

Figure 2: Minimally parameterized model – Ratios between standard deviations and mean parameter estimates

In the first case the corresponding parameterization of the first subsystem is of the type

$$(z^3 - \alpha_{113} z^2 - \alpha_{112} z - \alpha_{111}) y_1(t) - (\alpha_{122} z + \alpha_{121}) y_2(t) = (\beta_{113} z^2 + \beta_{112} z + \beta_{111}) u(t) \tag{24}$$

and has one extra parameter with respect to the minimal one. The associated values of the cost function for the parameters $\alpha_{113}, \alpha_{112}\, \alpha_{111}, \alpha_{122}, \alpha_{121}, \beta_{113}, \beta_{112}, \beta_{111}$ are reported in Fig. 3. The maximal value of the cost function jumps now to 141.28.

In the second case the structure of the model is of the type

$$(z^3 - \alpha_{113} z^2 - \alpha_{112} z - \alpha_{111}) y_1(t) - (\alpha_{123} z^2 + \alpha_{122} z + \alpha_{121}) y_2(t) = (\beta_{113} z^2 + \beta_{112} z + \beta_{111}) u(t) \tag{25}$$

and contains two extra parameters with respect to a minimal one. Fig. 4 reports the associated values of the cost function for the parameters $\alpha_{113}, \alpha_{112}\, \alpha_{111}, \alpha_{123}, \alpha_{122}, \alpha_{121}, \beta_{113}, \beta_{112}, \beta_{111}$. It shows two peaks with values equal to 93.08 and 98.52.

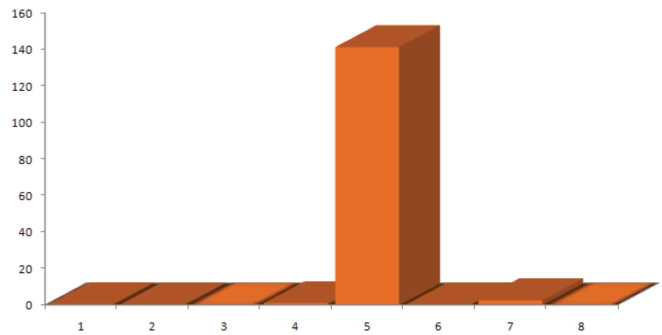

Figure 3: Model with one extra parameter – Ratios between standard deviations and mean parameter estimates

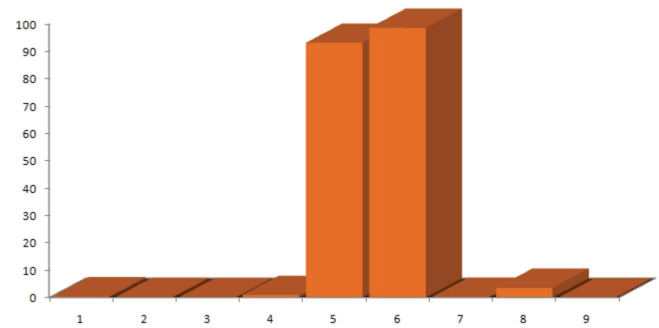

Figure 4: Model with two extra parameters – Ratios between standard deviations and mean parameter estimates

These simulations show that the use of equivalent but not minimally parameterized models leads to a remarkably larger dispersion of the estimated parameter values. It is of interest also to note that the larger dispersions concern more specifically the extra parameters introduced into the model.

5 Conclusions

This paper has considered the problem of the dispersion of the parameters in the identification of dynamic models and has shown, by means of Monte Carlo simulations, that non minimal parameterizations lead to dispersions larger than those associated with minimal ones. These results can be exploited when models to be identified for fault detection and localization must be selected.

The simulations have concerned simple ARX processes but quite similar results can be obtained also for more complex stochastic environments.

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