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MATHEMATICAL MODELLING OF FOREST FIRE FRONT EVOLUTION: WAVE APPROACH

J. Glasa[1], L. Halada[1]
[1]Institute of Informatics, Slovak Academy of Sciences, Bratislava, Slovakia

Corresponding author: J. Glasa, Institute of Informatics, Slovak Academy of Sciences
Dubravska cesta 9, 84507 Bratislava, Slovakia, jan.glasa@savba.sk

Abstract. In the paper a new updated analysis of wave approach to mathematical modelling of the forest fire front evolution in time is summarized. Mathematical foundations of elliptical fire spread model are described focusing on new original consequences followed from the knowledge of classical envelope theory of sets of curves used for the description of fire front evolution based on Huygens' principle. This approach allows us to formulate some essential assumptions of the model in original way and then to contribute to better understanding certain limitations of the model. The new approach to the model derivation enables to derive explicit formulae for the envelope of local elliptical fires ignited at a given starting fire front which form a new fire front after a given short time period. This procedure of the model derivation differs from the one known in the literature in which a linear transformation of co-ordinate axes which transforms ellipses into circles is necessary to utilize specific geometrical properties of points lying on common tangent line of two circles. The new approach avoids the necessity of this transformation for the derivation of the model. The knowledge of mathematical model of the fire spread under variable topographic, meteorological and fuel conditions is significant for effective implementation of the model, its correct use, and proper interpretation of simulation results in real conditions. The deeper analysis of the model can provide benefit for the readers engaged in the use of the computer simulation systems based on this model developed for the forest fire management purposes.

1 Introduction

Forest fires belong to destructive natural phenomena which have not been sufficiently described because of their complexity and still inadequate knowledge about the processes determining behaviour of the phenomenon, as well as because of a huge amount of data and serious difficulties with their extraction and gathering. Every year, forest fires cause large economical and environmental damages and devastate badly the landscape scenery, vegetation and eco-systems. They threaten environment, property and people's lives, and block significant human resources.

Advances in computers and information technologies stimulate research and development of advanced software tools capable to support decisions of fire management and fire fighting [6]. Mathematical modelling of forest fire spread plays a key role in existing decision support systems developed for planning, management and coordination of fire fighting activities and fire prevention. Such systems are capable to simulate the evolution of forest fire front in time and predict the spatial and temporal behaviour of forest fires (direction and rate of fire spread). They can quantify and display various fire characteristics such as the fireline intensity, flame length, etc. for the purposes of the fire effects evaluation. Computer fire simulation systems can be used in the fire prevention and planning. They allow to simulate various fire scenarios in a certain region under different conditions to test the fire management response for a fire event. They can be used as means for analysis of the effectiveness of different types of suppression strategies and tactics taking into account existing fire fighting infractructure and specific conditions in the region (road network, water sources, location of fire brigades, etc.). They allow to reconstruct past forest fire events during the post-suppression stage. Such systems can be useful for better understanding the circumstances of fatal losses of human lives during past forest fires [41, 42, 43, 22]. The fire behaviour predicting systems can be used for operational purposes. However, the current use of computer simulation for decision support purposes of forest fire management is still generally little in the European Union [47].

The knowledge of mathematical models of the fire spread implemented in existing simulation systems is significant not only for better understanding the models themselves, their effective implementation and correct use under variable topographic, meteorological and fuel conditions, but also for a proper interpretation of simulation results under real conditions. The fire spread in time is often described by a set of sophisticated rules for local fire spread obtained by averaging the typical fire behaviour observed during wildland and/or experimental fires or by a system of differential equations representing relations between relevant physical quantities influencing the fire behaviour. From different points of view, existing forest fire spread models can be classified as *stochastic* and *deterministic* models [36, 8, 24], or *empirical, semi-empirical* and *physical* models [35, 12, 5, 45, 33, 29].

Stochastic models are based on fires observations where the fire spread rate is related to relevant burning parameters such as fuel type, fuel moisture, wind speed, etc. in a statistical way to predict a more probable fire behaviour from average conditions. However, the obtained empirical relations strongly depend on specific conditions from which the statistical analysis is performed. Such models are also referred to as *empirical* models.

Deterministic models are usually divided to semi-empirical and physical models. *Semi-empirical* models are based on a global energy balance expression and on the assumption that the energy, which is transfered to the unburned fuel, is proportional to the energy released by the combustion of the fuel. Many useful relations of such models were obtained by fitting to extensive laboratory and outdoor fire experiments. The simplicity of this approach allows even to develop effective operational tools such as for example BEHAVE and FARSITE for the forest fire simulation in real conditions. *Physical* models take into account one or several processes of energy transfer from the burning zone to the unburned fuel. The combustion process is described using the conservation laws of mass, momentum, species and energy, and utilizing the knowledge of computer fluid dynamics. The systems WFDS [13, 46], FIRETEC [27], FIRESTAR [11] and CAFME [30] are examples od such systems. In general, physical models lead to differential equation systems requiring sophisticated and time-consuming numerical calculations and advanced high-performance computing environments [31]. They respect the heterogeneity of fuel structurein very small spaces. Despite the space resolution can be in centimeters, the resolution is strictly limited by the simulation space where the process is modelled (in 3D environment). The calculations are limited to relatively small space having a calculation time measured in hours that is why the successful parallelization of the calculations is a challenge for future. The simulation of more extensive real forest fires applicable for operational purposes capable to be realized on currently available computer equipment is then generally restricted for semi-empirical and empirical models.

Due to describe the fire spread in heterogeneous conditions, the problem of mathematical description of *local fire spread* in homogeneous conditions (on a flat ground in uniform continuous non-spotting fuels involving constant wind velocity, moisture content and slope) has been investigated for several decades. The analytical approximation of the fire shape, which is most often used in the fire research literature, is an ellipse [21, 40] and is referred as a *local elliptical model* of the fire spread. Other approximations which appear in the literature are ”teardrops”, ”ovoids” and ”double ellipses” [40, 34, 1, 3, 38]. For the description of steady-state forest fire spread in heterogeneous conditions, two basic principles are often used, namely a *cellular* and *wave* approach. Both these approaches have been studied intensively providing several successful implementations useable even for operational purposes such as for example FIRESTATION [28] and FARSITE [14] systems.

The idea of using the wave approach for prediction of a fire front evolution appeared in the literature about thirty years ago. The first method based on the wave principle, the radial fire propagation model [39], was developed in 1975. It used gridded weather inputs and rasterized topography of landscape and fuels to achieve a reasonable approximation of the observed fire growth. The concept of *Huygens' principle* and related terminology was brought into the fire literature in 1982, when a graphical fire prediction technique based on Huygens' principle was developed [4]. Other fire growth prediction techniques based on Huygens' principle were developed for purposes of fire growth simulation and visualization [15].

This paper is motivated by studies [4, 37] on Huygens‘ principle of wave propagation applied on the forest fire front evolution in time which are based on the assumption of local elliptical fire spread.

Each point on a starting fire front at a given time t can be considered as an ignition point of a small local fire which causes burning out of the area of elliptical shape at time $t+dt$. Assuming that each such an ellipse is defined by burning conditions at its generating ignition point, the resulting fire front at time $t+dt$ can be defined by the envelope of all the ellipses (*Huygens' principle* for the fire spread [4]).

The *system of differential equations* for description of the global steady-state forest fire spread in time based on Huygens' principle was derived analytically by G. D. Richards [37]. His approach has become the basis for several successful software systems for simulation of forest fire spread under real conditions. The system of differential equations was obtained by a special transform (rotation and comprimation of axes of co-ordinate system) which allowed to transform the ellipses into circles and then to utilize some specific geometrical properties of points lying on a common tangent line of two circles. This transform corresponds to specific additional assumptions on semi-axes of the corresponding ellipses and determined the final form of the equations derived [19].

In this paper we derive the model without the use of the transform used by Richards applying the knowledge of classical theory of envelopes of curves sets. This approach to the model derivation allows us to explain some essential assumptions of the model in original way and to better understand certain limitations of the model and its existing implementations and makes its further generalizations possible.

2 Richards' elliptical forest fire spread model

The fundamental assumption adopted in this section is the local elliptical fire spread in homogeneous conditions (on perfectly flat ground covered by homogeneous non-spotting fuel; constant wind) observed experimentally. This means that within time dt the border of burning area ignited in a point has the shape of ellipse with semi-axes adt and bdt where the centre of the ellipse is shifted by cdt in the wind direction [2, 4] (see Fig. 1).

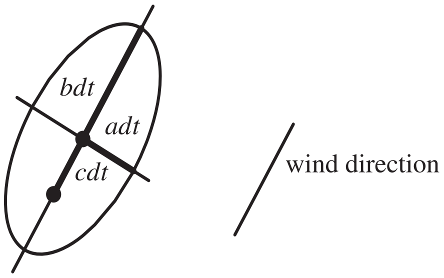


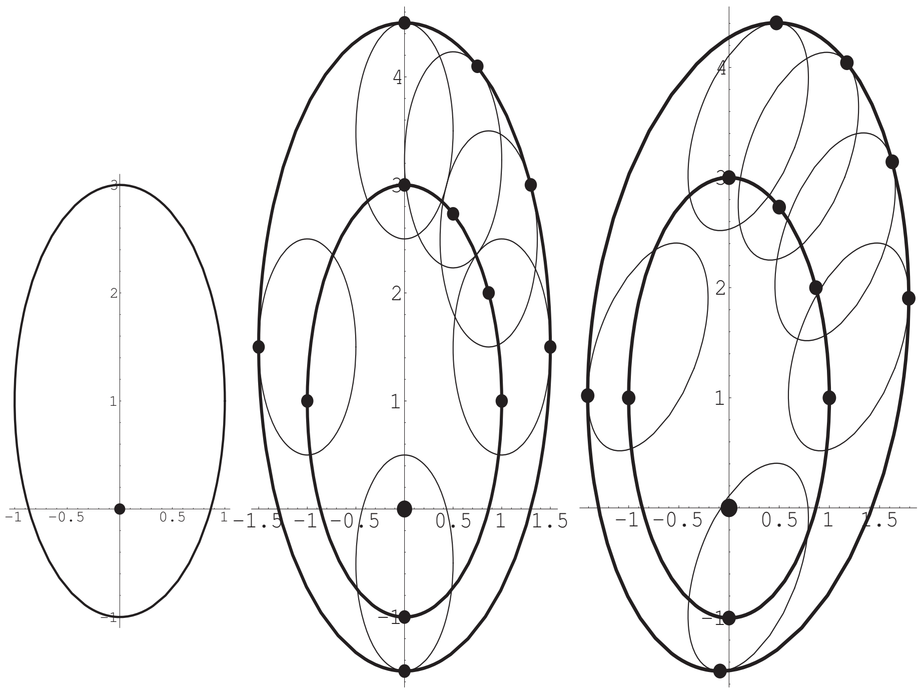
Figure 1: Local elliptical fire spread on perfectly flat ground covered by homogeneous fuel with constant wind [19].

In the following, let a *starting fire front* at time t, which borders the burning area at time t, be represented parametrically by a given simply closed planar curve $(x(\phi,t), y(\phi,t))$, where ϕ is a parameter dependent on the curve parametrization. Similarly, let a *new fire front* at time $t+dt$ be represented by a curve $(x(\phi,t+dt), y(\phi,t+dt))$. When both the curves $(x(\phi,t), y(\phi,t))$ and $(x(\phi,t+dt), y(\phi,t+dt))$ are known, the change of fire front position in time can be calculated for all values of the parameter ϕ as follows:

$$x_t(\phi,t) = \lim_{dt\to 0} \frac{x(\phi,t+dt) - x(\phi,t)}{dt},$$

$$y_t(\phi,t) = \lim_{dt\to 0} \frac{y(\phi,t+dt) - y(\phi,t)}{dt}.$$

Thus, we have the rate of change in time of co-ordinates of points lying on the fire front, as well as the system of differential equations, which describes this process [19].



(i) (ii) (iii)

Figure 2: Huygens' principle for the forest fire spread in time: (i) fire front at time t with the ignition point in the origin of co-ordinate system and homogeneous burning conditions given by $a = 1, b = 2, c = 1$; (ii-iii) starting fire front at time t, system of ellipses which represents homogeneous burning conditions along the starting fire front given by $a = 1/2, b = 1, c = 1/2$ and the corresponding new fire front at time $t+dt$. The change of wind direction $\theta = 0$ and $\theta = -\pi/8$ is shown on (ii) and (iii), respectively [19].

In the sequel, let the wind direction at time t be changed from the original direction in y-axis by an angle θ of the clockwise orientation and let it be constant in the whole burning period $[t, t+dt]$. Each point on the starting fire front can be then considered as the beginning of a local fire of elliptical shape. At time $t+dt$ such ellipses have the semi-axes $a(\phi,t)dt$ and $b(\phi,t)dt$ and the shift in the wind direction by $c(\phi,t)dt$ according to their ignition points. Let the functions $a(\phi,t)$, $b(\phi,t)$ and $c(\phi,t)$ be given by the burning conditions at the ignition points. According to

Huygens' principle, the new fire front at time $t+dt$, which borders the whole burning area at time $t+dt$, is defined by the envelope of the system of ellipses (see Fig. 2).

In order to utilize geometrical properties of points lying on a common tangent line of two circles, Richards introduced the transform [19]

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} b/a & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} C & -S \\ S & C \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = T \begin{bmatrix} x \\ y \end{bmatrix}, \tag{1}$$

where X,Y and x,y are co-ordinates in the transformed and original co-ordinate system, respectively, and C = cos θ, S = sin θ. The transform T rotates the axes of the co-ordinate system by the angle θ and re-scales the x-axis by a quotient b/a. By such a procedure, the same direction of the wind and the Y-axis, as well as the transform of ellipses into circles is achieved. The use of this co-ordinate transform is justified by observations from extensive fire experiments that the quotient of the parameters a,b for a given type of continuous non-spotting fuel and constant wind and slope depends on the wind speed only [37, 2]. This means that the quotient a/b depends on the time only and is spatially independent, i.e., for arbitrary $d\phi$ it holds [19]:

$$\frac{a(\phi,t)}{b(\phi,t)} = \frac{a(\phi+d\phi,t)}{b(\phi+d\phi,t)}. \tag{2}$$

The transform T and the corresponding assumption (2) makes it possible to find an envelope of circles instead of that of ellipses, and then to obtain the envelope of ellipses by the inverse transform:

$$T^{-1} = \begin{bmatrix} C & S \\ -S & C \end{bmatrix} \begin{bmatrix} a/b & 0 \\ 0 & 1 \end{bmatrix}.$$

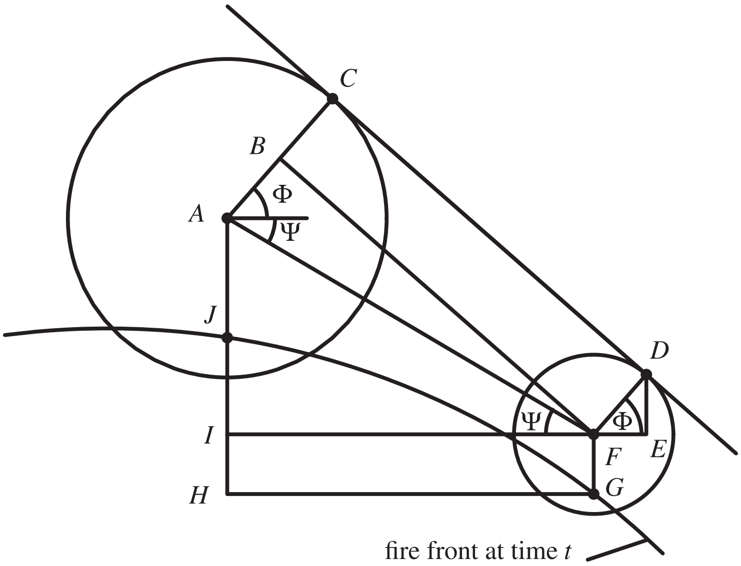


Figure 3: Two circles on transformed fire front (in co-ordinate system (X,Y)) [19].

Fig. 3 shows such two circles which represent a local fire spread from two points J and G lying on a starting fire front and correspond to the values (ϕ,t) and $(\phi+d\phi,t)$ (in new co-ordinate system). For points and distances given in Fig. 3 as $d\phi \to 0$, it holds [37]:

$$G = (X(\phi,t),Y(\phi,t)),\ J = (X(\phi+d\phi,t),Y(\phi+d\phi,t)),$$

$$C = (X(\phi,t+dt),Y(\phi,t+dt)),\ D = (X(\phi+d\phi,t+dt),Y(\phi+d\phi,t+dt)),$$

$$AB = dtd\phi b_\phi(\phi,t),\ DF = dtb(\phi,t),\ FG = dtc(\phi,t),\ FI = -d\phi X_\phi(\phi,t),$$

$$AF = (AI^2+FI^2)^{1/2} = d\phi\{[dtc_\phi(\phi,t)+Y_\phi(\phi,t)]^2 + X_\phi(\phi,t)^2\}^{1/2},$$

$$AJ = dtc(\phi+d\phi,t) = dtc(\phi,t)+dtd\phi c_\phi(\phi,t),\ AC = dtb(\phi+d\phi,t) = dtb(\phi,t)+dtd\phi b_\phi(\phi,t),$$

where the lower index ϕ denotes a partial derivative with respect to ϕ, and

$$b(\phi+d\phi,t) = b(\phi,t)+d\phi b_\phi(\phi,t),\ c(\phi+d\phi,t) = c(\phi,t)+d\phi c_\phi(\phi,t),$$

$$X(\phi+d\phi,t) = X(\phi,t)+d\phi X_\phi(\phi,t), Y(\phi+d\phi,t) = Y(\phi,t)+d\phi Y_\phi(\phi,t).$$

It follows from Fig. 3 that it holds [37]:

$$X[D] = X(\phi, t+dt) = X(\phi,t) + EF = X(\phi,t) + dtb(\phi,t)\cos(\Phi+\Psi-\Psi) =$$
$$= X(\phi,t) + dtb(\phi,t)[\cos(\Phi+\Psi)\cos\Psi + \sin(\Phi+\Psi)\sin\Psi], \tag{3}$$
$$Y[D] = Y(\phi,t+dt) = Y(\phi,t) + DE + FG = Y(\phi,t) + dtb(\phi,t)\sin(\Phi+\Psi-\Psi) + dtc(\phi,t) =$$
$$= Y(\phi,t) + dtb(\phi,t)[\sin(\Phi+\Psi)\cos\Psi - \cos(\Phi+\Psi)\sin\Psi] + dtc(\phi,t). \tag{4}$$

Since the line CD is the tangent line to both circles, if $d\phi \to 0$, then $C \to D$ and the line CD approaches the envelope of both circles, and the co-ordinates of point D approach the co-ordinates of $(X(\phi,t+dt), Y(\phi,t+dt))$. Therefore, after the substitution of the relations

$$\cos(\Phi+\Psi) = AB/AF,\ \sin(\Phi+\Psi) = \frac{(AF^2 - AB^2)^{1/2}}{AF},\ \cos\Psi = FI/AF,\ \sin\Psi = \frac{(AF^2 - FI^2)^{1/2}}{AF},$$

which follow from Fig. 3 into (3)-(4), we can calculate as $d\phi \to 0$:

$$X(\phi,t+dt) = X(\phi,t) + P(\phi,t,dt),$$
$$Y(\phi,t+dt) = Y(\phi,t) + Q(\phi,t,dt),$$

where

$$P(\phi,t,dt) = dtb\frac{-X_\phi dtb_\phi + (dtc_\phi + Y_\phi)[(dtc_\phi + Y_\phi)^2 + X_\phi^2 - dt^2 b_\phi^2]^{1/2}}{(dtc_\phi + Y_\phi)^2 + X_\phi^2},$$

$$Q(\phi,t,dt) = dtb\frac{-[(dtc_\phi + Y_\phi)^2 + X_\phi^2 - dt^2 b_\phi^2]^{1/2} X_\phi - (dtc_\phi + Y_\phi)dtb_\phi}{(dtc_\phi + Y_\phi)^2 + X_\phi^2} + cdt,$$

and functions b, c, b_ϕ, c_ϕ, X_ϕ, Y_ϕ are given in (ϕ,t). These equations hold in the co-ordinate system (X,Y). Since it holds:

$$T^{-1}\begin{bmatrix} X(\phi,t+dt) \\ Y(\phi,t+dt) \end{bmatrix} = T^{-1}\begin{bmatrix} X(\phi,t) + P(\phi,t,dt) \\ Y(\phi,t) + Q(\phi,t,dt) \end{bmatrix} =$$
$$= \begin{bmatrix} x(\phi,t) \\ y(\phi,t) \end{bmatrix} + \begin{bmatrix} C & S \\ -S & C \end{bmatrix}\begin{bmatrix} a/b & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} P(\phi,t,dt) \\ Q(\phi,t,dt) \end{bmatrix},$$

we have:

$$\begin{bmatrix} x(\phi,t+dt) \\ y(\phi,t+dt) \end{bmatrix} - \begin{bmatrix} x(\phi,t) \\ y(\phi,t) \end{bmatrix} = \begin{bmatrix} C & S \\ -S & C \end{bmatrix}\begin{bmatrix} (a/b)P(\phi,t,dt) \\ Q(\phi,t,dt) \end{bmatrix},$$

and applying the limit process as $dt \to 0$ to both sides of the equation, we get partial derivatives with respect to t:

$$\begin{bmatrix} x_t(\phi,t) \\ y_t(\phi,t) \end{bmatrix} = \lim_{dt\to 0}\frac{1}{dt}\begin{bmatrix} C & S \\ -S & C \end{bmatrix}\begin{bmatrix} (a/b)P(\phi,t,dt) \\ Q(\phi,t,dt) \end{bmatrix}.$$

Note that these arrangements can only be done under the assumption (2) which means that the quotient b/a is not a function of parameter ϕ. Since

$$\begin{bmatrix} X_\phi \\ Y_\phi \end{bmatrix} = \begin{bmatrix} b/a & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} C & -S \\ S & C \end{bmatrix}\begin{bmatrix} x_\phi \\ y_\phi \end{bmatrix} = \begin{bmatrix} (b/a)(Cx_\phi - Sy_\phi) \\ Sx_\phi + Cy_\phi \end{bmatrix},$$

we obtain the following system of differential equations [37]:

$$x_t(\phi,t) = \frac{a^2\cos\theta(x_\phi\sin\theta + y_\phi\cos\theta) - b^2\sin\theta(x_\phi\cos\theta - y_\phi\sin\theta)}{[b^2(x_\phi\cos\theta - y_\phi\sin\theta)^2 + a^2(x_\phi\sin\theta + y_\phi\cos\theta)^2]^{1/2}} + c\sin\theta \tag{5}$$

$$y_t(\phi,t) = \frac{-a^2\sin\theta(x_\phi\sin\theta + y_\phi\cos\theta) - b^2\cos\theta(x_\phi\cos\theta - y_\phi\sin\theta)}{[b^2(x_\phi\cos\theta - y_\phi\sin\theta)^2 + a^2(x_\phi\sin\theta + y_\phi\cos\theta)^2]^{1/2}} + c\cos\theta. \tag{6}$$

The system (5)-(6) describes the spread of the steady-state forest fire on perfectly flat terrain covered by continuous non-spotting fuel where the local elliptical fire spread and the constant eccentricity of the ellipses are assumed. A finite difference solution of (5)-(6) was suggested in [37] including its inherent problems (such as cross-overs, regriding, cross-over clipping) and their solution. This model has been generalized for slopy terrain and has become a part of several useful fire management tools. For instance, the Richards' technique is employed in the GIS-based software tool FARSITE which is widely used especially in the U.S. for both operational and planning purposes. The system incorporates several other fire behaviour models including surface fire, crown fire, point-source fire acceleration, spotting, and fuel moisture calculation model. Its applicability for the fire simulation in European forests was investigated recently [25, 26, 7, 22].

3 New approach to the elliptical fire spread model derivation

A new approach to the elliptical fire spread model derivation based on classical envelope theory from differential geometry was suggested in [19]. The envelope of the system of ellipses was directly derived without the use of the transform of co-ordinate system. The system of differential equations describing the elliptical forest fire spread was then obtained from the envelope by limit process, similarly as in the previous section.

Assuming (2), the relation between functions $a(\phi,t)$ and $b(\phi,t)$, and their derivatives for the case of variable burning conditions can be expressed as follows [19]:

$$\frac{a_\phi(\phi,t)}{b_\phi(\phi,t)} = \frac{a(\phi,t)}{b(\phi,t)}. \tag{7}$$

Indeed, assuming that for values ϕ and $\phi + d\phi$ it holds:

$$a(\phi,t)/b(\phi,t) = m,\ a(\phi+d\phi,t)/b(\phi+d\phi,t) = m,\ \text{for } m \in R - \{0\},$$

we get:

$$a(\phi+d\phi,t) - a(\phi,t) = m(b(\phi+d\phi,t) - b(\phi,t)),$$

$$(a(\phi+d\phi,t) - a(\phi,t))/d\phi = m(b(\phi+d\phi,t) - b(\phi,t))/d\phi,\ d\phi \neq 0,$$

and by limit process as $d\phi \to 0$ we have $a_\phi(\phi,t) = mb_\phi(\phi,t)$, from which we get (7), because $m = a(\phi,t)/b(\phi,t)$.

Let us consider a local fire of elliptical shape at time $t+dt$ in the form:

$$\begin{bmatrix} x_E(\phi,t+dt) \\ y_E(\phi,t+dt) \end{bmatrix} = \begin{bmatrix} C & S \\ -S & C \end{bmatrix} \begin{bmatrix} a(\phi,t)dt\cos\alpha \\ b(\phi,t)dt\sin\alpha + c(\phi,t)dt \end{bmatrix},$$

where $\alpha \in [0,2\pi]$ is a parameter, the origin of co-ordinate system is in a fire ignition point and the main semi-axis of the ellipse is in the wind direction.

The system of ellipses, which correspond to different values of parameter α, can be expressed in the form [19]:

$$\begin{bmatrix} x_1(\phi,t+dt) \\ y_1(\phi,t+dt) \end{bmatrix} = \begin{bmatrix} x_E(\phi,t+dt) + x(\phi,t) \\ y_E(\phi,t+dt) + y(\phi,t) \end{bmatrix}, \text{ i.e.,}$$

$$\begin{bmatrix} x_1(\phi,t+dt) \\ y_1(\phi,t+dt) \end{bmatrix} = \begin{bmatrix} x(\phi,t) + [a(\phi,t)dt\cos\alpha]C + [b(\phi,t)dt\sin\alpha + c(\phi,t)dt]S \\ y(\phi,t) - [a(\phi,t)dt\cos\alpha]S + [b(\phi,t)dt\sin\alpha + c(\phi,t)dt]C \end{bmatrix},$$

where t, dt and θ are constants, $a(\phi,t), b(\phi,t)$ and $c(\phi,t)$ are functions of parameter ϕ for a given value of t, ϕ and α are parameters, and x_1 and y_1 are variables. The system of ellipses is represented by 2-parameter system of curves with parameters ϕ and α, from which we get the 1-parameter system of curves:

$$\begin{aligned} &b(\phi,t)^2\{[x_1(\phi,t+dt) - x(\phi,dt)]C - [y_1(\phi,t+dt) - y(\phi,t)]S\}^2 + \\ &+a(\phi,t)^2\{[x_1(\phi,t+dt) - x(\phi,dt)]S + [y_1(\phi,t+dt) - y(\phi,t)]C - c(\phi,t)dt\}^2 - \\ &-a(\phi,t)^2 b(\phi,t)^2 dt^2 = 0, \end{aligned} \tag{8}$$

eliminating the parameter α by standard procedure (for more details see [19]). Formally, (8) can be written as

$$F(x_1(\phi,t+dt), y_1(\phi,t+dt), \phi) = 0,$$

where t, dt, θ are constants, $a(\phi,t), b(\phi,t), c(\phi,t)$ are functions of parameter ϕ for a given value of t, ϕ is parameter, and x_1, y_1 are variables.

Let the first partial derivative of F with respect to ϕ is equal to zero:

$$\begin{aligned} &b(\phi,t)b_\phi(\phi,t)\{[x_1(\phi,t+dt) - x(\phi,t)]C - [y_1(\phi,t+dt) - y(\phi,t)]S\}^2 + \\ &+b(\phi,t)^2[-x_\phi(\phi,t)C + y_\phi(\phi,t)S]\{[x_1(\phi,t+dt) - x(\phi,t)]C - [y_1(\phi,t+dt) - y(\phi,t)]S\} + \\ &+a(\phi,t)a_\phi(\phi,t)\{[x_1(\phi,t+dt) - x(\phi,t)]S + [y_1(\phi,t+dt) - y(\phi,t)]C - c(\phi,t)dt\}^2 + \\ &+a(\phi,t)^2[-x_\phi(\phi,t)S - y_\phi(\phi,t)C - c_\phi(\phi,t)dt]\{[x_1(\phi,t+dt) - x(\phi,t)]S + \\ &+[y_1(\phi,t+dt) - y(\phi,t)]C - c(\phi,t)dt\} - dt^2[a(\phi,t)a_\phi(\phi,t)b(\phi,t)^2 + b(\phi,t)b_\phi(\phi,t)a(\phi,t)^2] = 0. \end{aligned} \tag{9}$$

The solution of the system of two equations (8)-(9) of two variables x_1, y_1 represents a *discriminant curve* of the system (8) of ellipses. For simplicity, let us write individual functions more simply without the points $(\phi,t),(\phi,t+dt)$ in which they are defined, and let us introduce the following substitution:

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 - x \\ y_1 - y - cdt \end{bmatrix}. \tag{10}$$

Then, the system of equations (8)-(9) becomes:

$$b^2(x_2C - y_2S)^2 + a^2(x_2S + y_2C - cdt)^2 - a^2b^2dt^2 = 0, \tag{11}$$

$$\begin{aligned} &bb_\phi(x_2C - y_2S)^2 + b^2(x_2C - y_2S)(-x_\phi C + y_\phi S) + \\ &+ aa_\phi(x_2S + y_2C - cdt)^2 + a^2(x_2S + y_2C - cdt)(-x_\phi S - y_\phi C - c_\phi dt) - dt^2(aa_\phi b^2 + bb_\phi a^2) = 0, \end{aligned} \tag{12}$$

and after application of relations (2) and (7) onto (12), we have:

$$b^2(x_2C - y_2S)^2 + a^2(x_2S + y_2C - cdt)^2 - a^2b^2dt^2 = 0, \tag{13}$$

$$\begin{aligned} &bb_\phi(x_2C - y_2S)^2 + b^2(x_2C - y_2S)(-x_\phi C + y_\phi S) + \frac{a^2b_\phi}{b}(x_2S + y_2C - cdt)^2 + \\ &+ a^2(x_2S + y_2C - cdt)(-x_\phi S - y_\phi C - c_\phi dt) - 2dt^2a^2b_\phi b = 0. \end{aligned} \tag{14}$$

After such simplification, the system (13)-(14) has two solutions:

$$\begin{aligned} x_2 = {} & dtcS + \frac{-a^2bb_\phi dt^2(c_\phi dtS + x_\phi)}{a^2(c_\phi dt + Sx_\phi + Cy_\phi)^2 + b^2(Cx_\phi - Sy_\phi)^2} \pm \\ & \pm dt\frac{[a^2C(c_\phi dt + Sx_\phi + Cy_\phi) - b^2S(Cx_\phi - Sy_\phi)][b^2(Cx_\phi - Sy_\phi)^2 + a^2(c_\phi dt + Sx_\phi + Cy_\phi)^2 - a^2b_\phi^2dt^2]^{1/2}}{a^2(c_\phi dt + Sx_\phi + Cy_\phi)^2 + b^2(Cx_\phi - Sy_\phi)^2}, \\ y_2 = {} & dtcC + \frac{-a^2bb_\phi dt^2(c_\phi dtC + y_\phi)}{a^2(c_\phi dt + Sx_\phi + Cy_\phi)^2 + b^2(Cx_\phi - Sy_\phi)^2} \mp \\ & \mp dt\frac{[a^2S(c_\phi dt + Sx_\phi + Cy_\phi) + b^2C(Cx_\phi - Sy_\phi)][b^2(Cx_\phi - Sy_\phi)^2 + a^2(c_\phi dt + Sx_\phi + Cy_\phi)^2 - a^2b_\phi^2dt^2]^{1/2}}{a^2(c_\phi dt + Sx_\phi + Cy_\phi)^2 + b^2(Cx_\phi - Sy_\phi)^2}, \end{aligned}$$

which correspond to the outer and inner envelope of the system of ellipses. Using (10), we get the equations for the discriminant curve of the system of ellipses as follows:

$$\begin{aligned} x_1 = {} & x + dtcS + dt^2\frac{-a^2bb_\phi(c_\phi dtS + x_\phi)}{a^2(c_\phi dt + Sx_\phi + Cy_\phi)^2 + b^2(Cx_\phi - Sy_\phi)^2} \pm \\ & \pm dt\frac{[a^2C(c_\phi dt + Sx_\phi + Cy_\phi) - b^2S(Cx_\phi - Sy_\phi)][b^2(Cx_\phi - Sy_\phi)^2 + a^2(c_\phi^2 dt + Sx_\phi + Cy_\phi)^2 - a^2b_\phi^2dt^2]^{1/2}}{a^2(c_\phi dt + Sx_\phi + Cy_\phi)^2 + b^2(Cx_\phi - Sy_\phi)^2} \end{aligned} \tag{15}$$

$$\begin{aligned} y_1 = {} & y + dtcC + dt^2\frac{-a^2bb_\phi(c_\phi dtC + y_\phi)}{a^2(c_\phi dt + Sx_\phi + Cy_\phi)^2 + b^2(Cx_\phi - Sy_\phi)^2} \mp \\ & \mp dt\frac{[a^2S(c_\phi dt + Sx_\phi + Cy_\phi) + b^2C(Cx_\phi - Sy_\phi)][b^2(Cx_\phi - Sy_\phi)^2 + a^2(c_\phi^2 dt + Sx_\phi + Cy_\phi)^2 - a^2b_\phi^2dt^2]^{1/2}}{a^2(c_\phi dt + Sx_\phi + Cy_\phi)^2 + b^2(Cx_\phi - Sy_\phi)^2}. \end{aligned} \tag{16}$$

Note that the solution of system (13)-(14) leads to the solution of an equation of the fourth order in which, applying relations (2) and (7), the coefficients for two highest powers are equal to zero (they disappear). Therefore, the solved equation of the fourth order is quadratic having two solutions. According to analysis of conditions for singular points, which are not fulfilled for any point of the discriminant curve (15)-(16) of the system of ellipses, the equations (15)-(16) represent the *envelope of system of ellipses* [19]:

$$\begin{aligned} x(\phi,t+dt) = {} & x(\phi,t) + dtcS + dt^2\frac{-a^2bb_\phi(c_\phi dtS + x_\phi)}{a^2(c_\phi dt + Sx_\phi + Cy_\phi)^2 + b^2(Cx_\phi - Sy_\phi)^2} \pm \\ & \pm dt\frac{[a^2C(c_\phi dt + Sx_\phi + Cy_\phi) - b^2S(Cx_\phi - Sy_\phi)][a^2(c_\phi dt + Sx_\phi + Cy_\phi)^2 + b^2(Cx_\phi - Sy_\phi)^2 - a^2b_\phi^2dt^2]^{1/2}}{a^2(c_\phi dt + Sx_\phi + Cy_\phi)^2 + b^2(Cx_\phi - Sy_\phi)^2}, \end{aligned} \tag{17}$$

$$\begin{aligned} y(\phi,t+dt) = {} & y(\phi,t) + dtcC + dt^2\frac{-a^2bb_\phi(c_\phi dtC + y_\phi)}{a^2(c_\phi dt + Sx_\phi + Cy_\phi)^2 + b^2(Cx_\phi - Sy_\phi)^2} \mp \\ & \mp dt\frac{[a^2C(c_\phi dt + Sx_\phi + Cy_\phi) + b^2S(Cx_\phi - Sy_\phi)][a^2(c_\phi dt + Sx_\phi + Cy_\phi)^2 + b^2(Cx_\phi - Sy_\phi)^2 - a^2b_\phi^2dt^2]^{1/2}}{a^2(c_\phi dt + Sx_\phi + Cy_\phi)^2 + b^2(Cx_\phi - Sy_\phi)^2}. \end{aligned} \tag{18}$$

The system (17)-(18) represents the outer and inner envelopes of the system of ellipses, where the outer envelope corresponds to the new fire front which is to be found. Note that the envelope existence conditions have the form [19]:

$$a^2(c_\phi dt + Sx_\phi + Cy_\phi)^2 + b^2(Cx_\phi - Sy_\phi)^2 - a^2 b_\phi^2 dt^2 \geq 0,\ a^2(c_\phi dt + Sx_\phi + Cy_\phi)^2 + b^2(Cx_\phi - Sy_\phi)^2 \neq 0.$$

The derivation of the system of differential equations from (17)-(18), which describes the fire growth, is straightforward by limit process as $dt \to 0$, i.e.

$$x_t(\phi,t) = \lim_{dt\to 0} \frac{x(\phi,t+dt) - x(\phi,t)}{dt},$$

$$y_t(\phi,t) = \lim_{dt\to 0} \frac{y(\phi,t+dt) - y(\phi,t)}{dt}.$$

After the substitution of (17)-(18) we get the formulae which correspond to the equations (5)-(6) derived by Richards in [37] under the same assumptions.

As it has been shown in our previous paper [16], where the Richards' approach was re-formulated by the use of the apparatus of envelope theory *using* Richards' transform of co-ordinate axes, the resulting formulae are the same (under the same assumptions). Using the standard procedure for the envelope derivation, the model is now derived *without* the use of the Richards' transform of the co-ordinate system. Assuming (2), we show the equivalence between both presented procedures, because the substantial reduction of complexity of the resulting formulae is achieved (the same as in the Richards' approach).

The application of Huygens' principle for modelling the forest fire spread in which the new fire front is represented by the envelope of system of ellipses is illustrated in Figs. 4-5. The change of burning conditions along the starting fire front is shown in Fig. 4. It is represented by the functions $a(\phi,t)$, $b(\phi,t)$ for which $2a(\phi,t) = b(\phi,t)$ holds, therefore they fulfil the property (2) of the model.

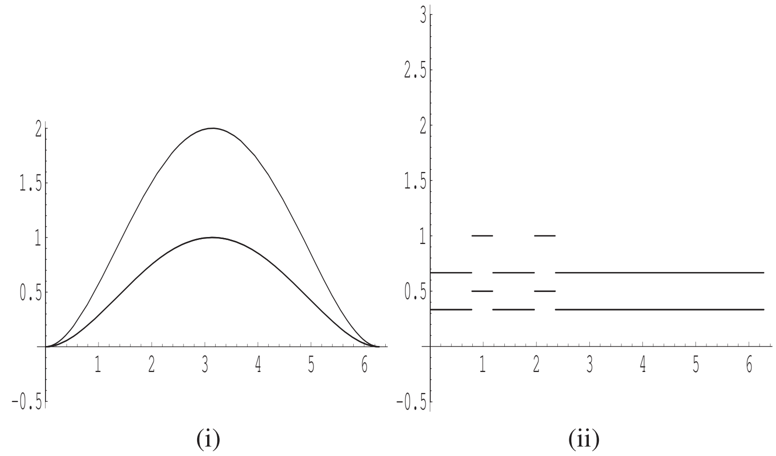
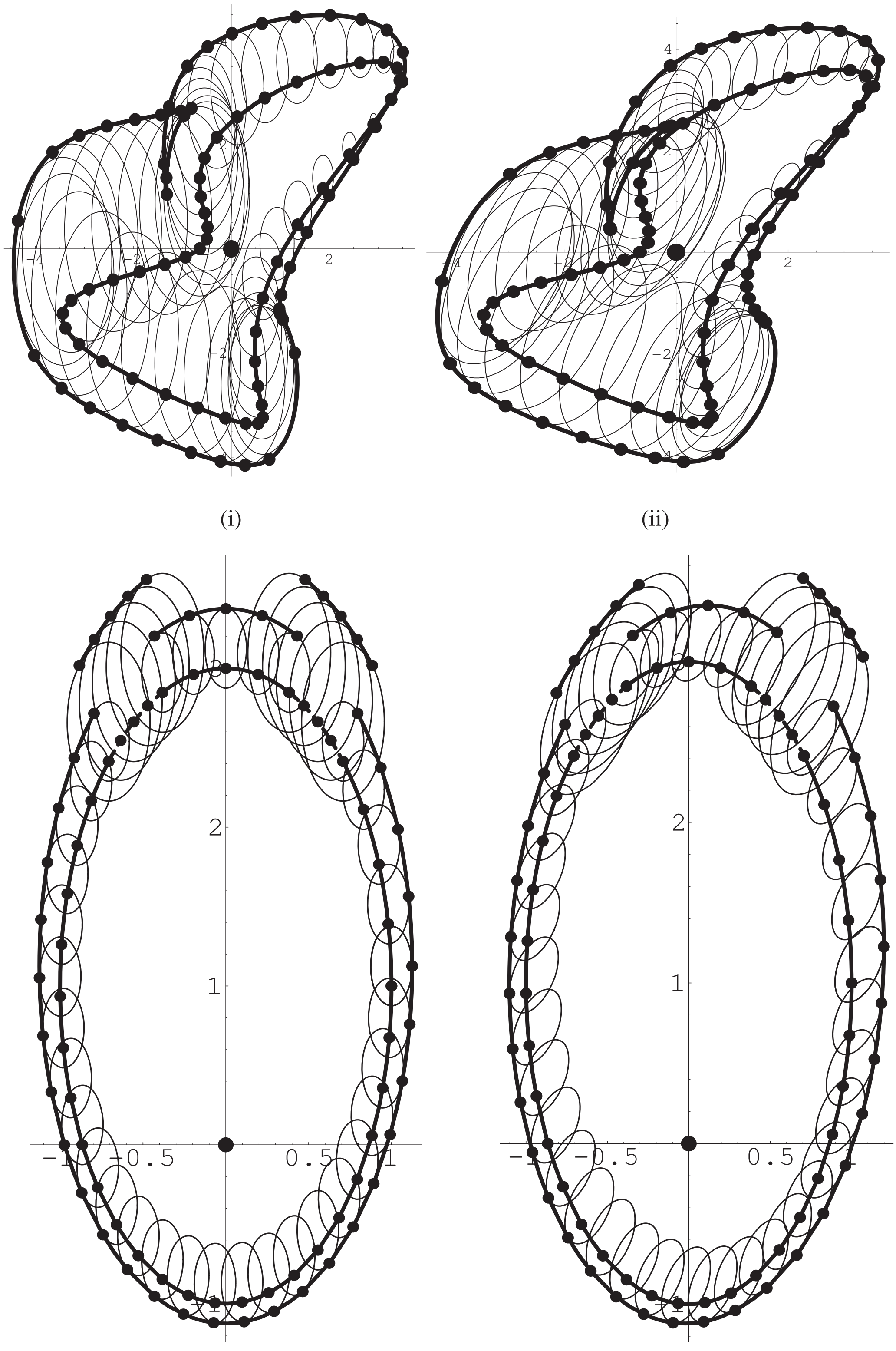


Figure 4: Two examples of functions $a(\phi,t)$, $b(\phi,t)$ representing the change of burning conditions along starting fire front for which it holds $2a(\phi,t) = b(\phi,t)$: Landau kernel (i) and step function (ii). The functions represent the continuous (i) and discontinuous (ii) change of burning conditions.

Fig. 5 demonstrates the fire spread under variable burning conditions (along an elliptical as well as a non-trivial non-convex started fire front) which determine the shape of 45 selected ellipses representing secondary fires in 45 ignition points lying on the starting fire front and the new fire front defined by the envelope of these ellipses. We use two examples of the starting fire front: an ellipse (simple convex shape) and a "general" curve (non-convex shape). The illustrated fire spreads correspond to the change of burning conditions given in Fig. 4 and the change of wind direction $\theta = 0$ and $\theta = -\pi/8$. Fig. 5 also illustrates different kinds of singularities and internal loops of resulting discriminant curve which can appear during forest fire spread simulation under real burning conditions which appear even in the case when the functions a, b fulfil the property (2).

4 Conclusion

In the paper, mathematical foundations of the elliptical model of the steady-state forest fire spread are summarized. The model is based on Huygens' wave principle assuming that each point on the starting fire front becomes the ignition source of a local fire of elliptical shape and the new fire front is defined by the envelope of these ellipses. Using the same assumptions as was used by Richards to derive the model, we apply standard procedure for the envelope derivation to derive explicite formulae for the envelope of ellipses which forms the new fire front. The



(i)

(ii)

(iii)

(iv)

Figure 5: Description of the use of Huygens' principle for forest fire front evolution in time for variable burning conditions. The new fire front is calculated by explicit formulae derived using the new approach to the elliptical fire spread model derivation based on the knowledge of envelope theory of curves sets.

proposed procedure differs from the model derivation proposed by Richards and avoids the necessity of transforming the co-ordinate system used by Richards to transform ellipses to circles and to utilize specific geometrical properties of points lying on common tangent line of two circles. We get the same resulting formulae for the system of different equations and show the equivalence between both the procedures. We derive the new relation between parameters a, b representing the variability of burning conditions along the starting fire front and their spatial derivatives. The readers who are engaged in the research and use of computer simulation systems and models developed for the forest fire spread modelling and simulation can benefit from the deeper analysis of fundaments of the mathematical fire spread model investigated. The new approach to the elliptical fire spread model derivation allows to suggest further generalizations of the model because it is not so strictly dependent on the assumption of local fire spread. Moreover, the experimental results and corresponding methodology published in [14] are a challenge for future study to improve the formulation of fire front evolution in time. The fireline rotation model presented by D. X. Viegas [32, 44] together with tangential forces activated by convective fluxes across the fire front can be the way how to improve the standard semi-empirical fire spread models.

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