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A MODEL FOR A MILDLY-SEVERELY STAGED DISEASE

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Abstract. Mathematical epidemiology studies the infectious disease propagation among populations. The basic assumption consists in partitioning the whole population into a few classes, of which susceptible and infected individuals are compulsorily present in the model. Other classes could account for instance for latent individuals, i.e. those infected but in which the disease is still not able to spread to others, removed individuals, i.e. those who have been recognized as infectious and therefore quarantined, and so forth.

For many diseases that were considered fatal long time ago, but for which suitable cures were developed, a recrudescence is observed in these days, due to the fact that the infectious agents develop strains that are resistant to the administered drugs, when the latter perhaps are not assumed regularly or with the prescribed protocol. The goal of this paper is the study of the transmission of a disease which shows a weak, curable form, and a strong, possibly lethal stage. From the either one of the two stages individuals can move to the other one, this being the result of success or failure in the cures.

Our results show that if the net birth rate of the population is negative, the system collapses and the population is wiped out. However if the net birth rate is positive, there occur two possible equilibria: the origin now becomes unstable and an interior endemic equilibrium now arises. We have shown numerically that it bifurcates to originate stable limit cycles around it, for suitable choices of the parameter values. The influence of the model parameters is simulated and the key ones for its control are identified.

1 Introduction

Mathematical epidemiology studies the infectious disease propagation among populations. The basic assumption consists in partitioning the whole population into a few classes, of which susceptible and infected individuals are compulsorily present in the model. Other classes could account for instance for latent individuals, i.e. those infected but in which the disease is still not able to spread to others, removed individuals, i.e. those who have been recognized as infectious and therefore quarantined, and so forth.

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This paper is not a complete evaluation of the possible strategies in administration of the therapies of these diseases. This has been attempted elsewhere for the case of tuberculosis, [10]. The interested reader can also consult the bibliography of the paper [10] for further references on this theme. Rather, we focus here really on the possibility of having a mild form of a disease and a virulent, possibly lethal one, although in principle allowing the possibility of recovering from both and look only at the dynamics among these stages.

2 System equations

Let $S(t)$ denote the class of susceptibles, $W(t)$ denote the class of weak infectives and $V(t)$ denote the class of strong infectives. The total population is then $N(t) = S(t) + W(t) + V(t)$. The equations of model we propose are as follows

$$\begin{cases} \dot{S} = \rho\left(1 - \frac{1}{K}N\right)S - \beta SW - (\gamma_1 + \gamma_2)SV + \pi W + \nu V, \\ \dot{W} = -\lambda W + \beta SW + \gamma_1 SV - (\pi + m)W, \\ \dot{V} = \lambda W + \gamma_2 SV - \nu V - \mu V. \end{cases} \tag{1}$$

All parameters of the model are taken to be real and nonnegative, except for ρ which can be negative.

Susceptible individuals are the only ones able to reproduce and grow following a logistic model with population pressure constant K^{-1}, with net birth rate $\rho = b - m$, this being the difference between the birth b and the death m rates. They can catch the disease by means of contact with weak infectives with incidence β, and thus move to class W, but they also can come in contact with the strong infectives at rate $\gamma_1 + \gamma_2$. As a result of this intercourse

they can become weakly or strongly infected. Namely, the fraction γ_1 of them migrates to class W, the remaining one at rate γ_2 goes into class V.

Individuals that contracted the weak form of the disease can also worsen or recover. In the former case they move to class V, with rate λ. Those who recover return to class S with rate π. The weak infectives die with natural death rate m.

Finally, the strongly infected can also recover to pass to class S with rate ν. They also experience disease-related mortality, so that their cumulative death rate, including also the natural mortality, is denoted by μ.

Note that the model allows for direct transitions “down the line" from susceptibles to both kinds of infected, while it does not allow a partial recovery from the strong form of the disease back to its mild form, only a complete healing is envisaged, this leading a strongly infected back into the class of susceptibles. This makes sense, since we assume an SIS type of disease. In [10] more complicated recovery dynamics are considered.

The logistic model is rather complicated, so in what follows we will consider the following simplified model where susceptible individuals grow by following a Malthus model with birth and natural death rate ρ:

$$\begin{cases} \dot{S} = \rho S - \beta SW - (\gamma_1+\gamma_2)SV + \pi W + \nu V, \\ \dot{W} = -\lambda W + \beta SW + \gamma_1 SV - (\pi + m)W, \\ \dot{V} = \lambda W + \gamma_2 SV - \nu V - \mu V. \end{cases} \tag{2}$$

The paper is organized as follows. We begin by giving a boundedness result for the full system (1) and then we turn to a more thorough analysis of (2) in Sections 4 and 5. In the following Section we include the consideration of several particular cases of interest. Some thorough simulations are then discussed in Section 7. We conclude with some inferences on the experience made with this model.

3 Boundedness of the logistic model

In this section, we consider the logistic model (1) and show the boundedness of its solutions. To do this, we consider the total population $N = S+W+V$, and an arbitrary constant $\alpha > 0$, and find upon summation of the equations (1)

$$\dot{N} + \alpha N = -\frac{\rho}{K}S^2 + (\rho + \alpha - \frac{\rho}{K}W - \frac{\rho}{K}V)S + (\alpha - m)W + (\alpha - \mu)V \equiv P(S).$$

But we can rewrite the right hand side in the form $P(S) \equiv P_1(S) + P_2(S)$ where

$$P_1(S) = -\frac{\rho}{K}S^2 + (\rho+\alpha)S, \quad P_2(S) = -\frac{\rho}{K}(W-V)S + (\alpha - m)W + (\alpha - \mu)V. \tag{3}$$

Since in (3) the parabola can be bounded above by the value at its vertex $S_{max} = \frac{\rho+\alpha}{2\rho}K$ giving

$$P_1(S) \le P_1(S_{max}) = \frac{(\rho+\alpha)^2}{4\rho}K,$$

and if $\alpha < \min\{\mu, \nu\}$, also $P_2(S) \le 0$. It follows that

$$\dot{N} + \alpha N \le \frac{(\rho+\alpha)^2}{4\rho}K.$$

By applying Gronwall's inequality it then follows

$$N(t) \le C\, e^{-\alpha t} + \frac{(\rho+\alpha)^2}{4\rho\alpha}K \le H, \quad C, H \in \mathbf{R}$$

showing as desired the boundedness of the total population and therefore also of each one of its subpopulations.

4 Equilibrium points of the simplified model

The origin (0,0,0) is the only boundary equilibrium point. Note that if $\rho < 0$, then by adding all the equations (2) we find the dynamics of the whole population

$$\dot{N} = \rho S - mW - \mu V < 0, \tag{4}$$

indicating that the population vanishes, i.e. suggesting that the origin in this case is a globally asymptotically stable equilibrium.

From now on, we therefore focus on the model (2) assuming that ρ is positive. Searching for nontrivial equilibria, we find the points (S,W,V) with

$$W = \frac{\rho S(\mu+\nu-\gamma_2 S)}{m\mu+m\nu+\lambda\mu-m\gamma_2 S}\,, \qquad V = \frac{\lambda\rho S}{m\mu+m\nu+\lambda\mu-m\gamma_2 S}\,,$$

where S is determined by the two roots of the parabola

$$\Sigma(S) = \beta\gamma_2 S^2 - (m\gamma_2+\gamma_2\lambda+\beta\mu+\gamma_2\pi+\beta\nu+\gamma_1\lambda)S + (\lambda+\pi+m)(\mu+\nu),$$

i.e. explicitly

$$S_{1,2} = \frac{(\lambda+\pi+m)\gamma_2+\lambda\gamma_1+\beta(\mu+\nu) \mp \sqrt{\Delta}}{2\beta\gamma_2}.$$

Here,

$$\begin{aligned}\Delta &= [(\lambda+\pi+m)\gamma_2+\lambda\gamma_1-\beta(\mu+\nu)]\,(\lambda+\pi+m)\gamma_2 \\ &+ [\beta(\mu+\nu)-(\lambda+\pi+m)\gamma_2+\lambda\gamma_1]\,(\mu+\nu)\beta + [(\lambda+\pi+m)\gamma_2+\lambda\gamma_1-\beta(\mu+\nu)]\,\gamma_1\lambda.\end{aligned}$$

The roots are real if

$$\Delta > 0 \tag{5}$$

and since

$$\Sigma(0) > 0, \quad \Sigma'(0) = -(m\gamma_2+\gamma_2\lambda+\beta\mu+\gamma_2\pi+\beta\nu+\gamma_1\lambda) < 0,$$

both S_1 and S_2 are positive. To get feasibility of the equilibrium, we need to require positivity of all its components $S_{1,2}$, $W_{1,2}$ and $V_{1,2}$. This reduces to satisfying the inequality

$$0 < S_{1,2} < \frac{\mu+\nu}{\gamma_2}\,. \tag{6}$$

Upon verification, it turns out that only (S_1,W_1,V_1) is a feasible solution, while (S_2,W_2,V_2) is not acceptable. The feasibility condition (6) of (S_1,W_1,V_1) is equivalent to the following explicit condition in terms of the model parameters

$$\beta < \frac{\gamma_2(\lambda+\pi+m)+\lambda\gamma_1}{\mu+\nu}\,. \tag{7}$$

In summary, for $\rho > 0$, the model exhibits the equilibria

$$E_0 = (0,0,0), \quad E_1 = \left(S_1, \frac{\rho S_1(\mu+\nu-\gamma_2 S_1)}{m\mu+m\nu+\lambda\mu-m\gamma_2 S_1}, \frac{\lambda\rho S_1}{m\mu+m\nu+\lambda\mu-m\gamma_2 S_1}\right).$$

When $\rho < 0$ the model has no interior solutions as neither (S_1,W_1,V_1) nor (S_2,W_2,V_2) are feasible; as remarked above its only equilibrium point is the origin.

5 Stability of equilibria

5.1 The case $\rho < 0$

The characteristic equation of the Jacobian evaluated at the origin has the three negative eigenvalues $\sigma_1 = \rho$, $\sigma_2 = -\lambda-\pi-m$, and $\sigma_3 = -\mu-\nu$, so $E_0 = (0,0,0)$ is a locally asymptotically stable equilibrium point. In this case we have already seen that the total population always decreases to zero, so no other equilibrium can exist. Being the only one in this case, the origin is also globally asymptotically stable.

This can also be rigorously shown by using the total population size N as Lyapunov function L. Note indeed that $L(S,W,V) \equiv N \geq 0$ everywhere in the subset R_+ of the phase space, $R_+ \equiv \{(S,W,V) : S \geq 0, W \geq 0, V \geq 0\}$. Clearly $L(E_0) = 0$. From (4) it follows that $\dot{L} < 0$ in R_+. These properties ensure then that L is a suitable Lyapunov function, and global asymptotic stability of E_0 follows.

5.2 The case $\rho > 0$

To analyze the stabilty of equilibrium points, we need the Jacobian of (2)

$$J_{S,W,V} = \begin{pmatrix} \rho-\beta W-(\gamma_1+\gamma_2)V & -\beta S+\pi & -(\gamma_1+\gamma_2)S+\nu \\ \beta W+\gamma_1 V & -\lambda+\beta S-\pi-m & \gamma_1 S \\ \gamma_2 V & \lambda & \gamma_2 S-\mu-\nu \end{pmatrix}.$$

At E_0 the characteristic equation has the eigenvalues $\sigma_1 = \rho$, $\sigma_2 = -\lambda - \pi - m$, $\sigma_3 = -\mu - \nu$, from which it follows that E_0 is unstable.

Similarly, the characteristic equation evaluated at E_1 is

$$0 = D_1 \equiv \begin{vmatrix} \rho - \beta W_1 - (\gamma_1+\gamma_2)V_1 - \sigma & -\beta S_1 + \pi & -(\gamma_1+\gamma_2)S_1 + \nu \\ \beta W_1 + \gamma_1 V_1 & -\lambda + \beta S_1 - \pi - m - \sigma & \gamma_1 S_1 \\ \gamma_2 V_1 & \lambda & \gamma_2 S_1 - \mu - \nu - \sigma \end{vmatrix}.$$

Explicitly, it is the cubic equation

$$-\sigma^3 + C\sigma^2 + B\sigma + A = 0,$$

where the coefficients A, B e C are rather complicated functions of the parameters of the model and of the equilibria value, as follows

$$\begin{aligned} A = \frac{1}{m\mu + m\nu + \lambda\mu - m\gamma_2 S_1} \big(&\rho S_1^2 m^2\gamma_2^2 + \rho m\mu^2\pi + \rho m\nu^2\pi + \rho\lambda\mu\pi\nu 2\rho\lambda\mu\pi\gamma_2 S_1 - 2\rho\lambda^2\mu\gamma_1 S_1 \\ &-2\rho\lambda^2\mu\gamma_2 \S 1 + \rho\lambda\mu^2\pi - 2\rho m\nu\gamma_1 S_1\lambda + 3\rho\lambda\mu\beta S_1^2\gamma_2 - 2\rho\lambda\mu^2\beta S_1 - 2\rho\lambda\mu\beta S_1\nu + \rho m^2\nu^2 - 2\rho m\mu\gamma_1 S_1\lambda \\ &-2\rho m\nu^2\beta S_1 + 2\rho m\mu\pi\nu - 2\rho m\mu^2\beta S_1 - 4\rho m\mu\beta S_1\nu + \rho S_1^2 mgg\gamma_1\lambda + 3\rho m\mu\lambda\nu + 2\rho m\mu^2\lambda + 4\rho S_1^2 m\gamma_2\beta\nu \\ &+\rho S_1^2 m\gamma_2^2\pi - 2\rho S_1 m^2\gamma_2\mu + 4\rho S_1^2 m\gamma_2\beta\mu - 2\rho S_1 m\gamma_2\pi\mu - 2\rho S_1 m\gamma_2\pi\nu - 2\rho S_1 m^2\gamma_2\nu - 2\rho S_1 m\gamma_2\lambda\nu \\ &-2\rho S_1^3 m\gamma_2^2\beta + \rho m^2\mu^2 + \rho S_1^2 m\gamma_2^2\lambda - 4\rho S_1 m\gamma_2\lambda\mu + \rho\lambda^2\mu\nu + \rho\lambda^2\mu^2 + 2\rho m^2\mu\nu + \rho m\nu^2\lambda\big)\ , \end{aligned}$$

$$\begin{aligned} B = \frac{1}{m\mu + m\nu + \lambda\mu - m\gamma_2 S_1} (&-\beta S_1\rho\nu\lambda - 2\rho\lambda\mu\beta S_1 + \rho m\mu\pi + 2\rho m\mu\lambda + 3S_1 m\gamma_2\lambda\mu - \rho\mu\gamma_1 S_1\lambda \\ &-m\nu^2\lambda - m\nu^2\pi + \rho m^2\mu - 2m^2\mu\nu + S_1^3 m\gamma_2^2\beta - \lambda^2\mu^2 + 2S_1 m\gamma_2\pi\nu + 2S_1 m\gamma_2\lambda\nu - 2S_1^2 m\gamma_2\beta\nu - S_1^2 m\gamma_2^2\pi \\ &+2S_1 m^2\gamma_2\mu - 2S_1^2 m\gamma_2\beta\mu + 2S_1 m\gamma_2\pi\mu + 2S_1 m^2\gamma_2\nu + m\mu^2\beta S_1 + m\nu^2 * \beta S_1 - S_1^2 m^2\gamma_2^2 + m\mu\gamma_1 S_1\lambda \\ &-2m\mu\pi\nu + 2m\mu\beta S_1\nu - S_1^2 m\gamma_2\gamma_1\lambda + m\nu\gamma_1 S_1\lambda - \lambda\mu\beta S_1^2\gamma_2 + \lambda\mu^2\beta S_1 + \lambda\mu\beta S_1\nu - \lambda\mu^2\pi - \lambda\mu\pi\nu \\ &+\lambda^2\mu\gamma_1 S_1 + \lambda^2\mu\gamma_2 S_1 - m\mu^2\pi - \rho S_1^3\gamma_2^2\beta + 2\rho S_1^2\gamma_2\beta\mu + 2\rho S_1^2\gamma_2\beta\nu - 2\rho\mu\beta S_1\nu - \rho\mu^2\beta S_1 - \rho\nu^2\beta S_1 \\ &+\rho m\mu^2 + \rho m\nu^2 + \rho\lambda\mu^2 - 2\rho S_1 m\gamma_2\mu - 2\rho S_1 m\gamma_2\nu + \rho\lambda\mu\nu + 2\rho m\mu\nu + \rho S_1^2 m\gamma_2^2 - 2m\mu^2\lambda - \lambda^2\mu\nu \\ &-3m\mu\lambda\nu + \lambda\mu\pi\gamma_2 S_1 + \rho\lambda^2\mu - m^2\nu^2 + 2\rho S_1^2 m\gamma_2\beta - \rho S_1 m\gamma_2\pi - 2\rho S_1 m\gamma_2\lambda - 2\rho m\mu\beta S_1 - 2\rho m\nu\beta S_1 \\ &+2\beta S_1^2\rho\gamma_2\lambda + \rho m\nu\lambda + \rho m\nu\pi + \rho\lambda\mu\pi - S_1\lambda^2\rho\gamma_1 - S_1\lambda\rho\gamma_2\pi - S_1\lambda\rho\gamma_1 m - S_1\lambda^2\rho\gamma_2 - \rho\nu\gamma_1 S_1\lambda \\ &-2\rho\lambda\mu\gamma_2 S_1 - m^2\mu^2 + \rho m^2\nu - \rho S_1 m^2\gamma_2 - S_1^2 m\gamma_2^2\lambda\big)\,, \end{aligned}$$

and

$$\begin{aligned} C = \frac{1}{m\mu + m\nu + \lambda\mu - m\gamma_2 S_1} \big(&\lambda\mu\gamma_2 S_1 - \rho S_1 m\gamma_2 - \beta S_1\rho\mu - S_1\lambda\rho\gamma_1 - S_1\lambda\rho\gamma_2 + \beta S_1^2\rho\gamma_2 - S_1^2 m\gamma_2^2 \\ &-m\mu\pi - 2m\mu\nu + S_1 m^2\gamma_2 - m\mu^2 + S_1 m\gamma_2\lambda - \beta S_1\rho\nu - S_1^2 m\gamma_2\beta + 2S_1 m\gamma_2\mu \\ &+S_1 m\gamma_2\pi + m\mu\beta S_1 + m\nu\beta S_1 + 2S_1 m\gamma_2\nu + \rho\lambda\mu + \rho m\mu - m^2\mu - \lambda\mu^2 - m\nu^2 - m^2\nu \\ &- +\lambda\mu\beta S_1 + \rho m\nu - m\nu\lambda\lambda^2\mu - m\nu\pi - \lambda\mu\pi - 2m\mu\lambda - \lambda\mu\nu\big)\,. \end{aligned}$$

To find the three eigenvalues, we can intersect the linear function $B\sigma + A$ with the cubic $\sigma^3 - C\sigma^2$, which has a double root at the origin and the remaining one at $s = C \in \mathbf{R}$.

For $C > 0$ we have at least an intersection with the straight line $B\sigma + A$, real positive or imaginary with real positive part, guaranteeing thus instability. Further, when $C < 0$ the intersections with the straight line $B\sigma + A$ have negative real parts if we impose $A, B < 0$.

In order that a Hopf-bifurcation occurs, we need to have immaginary pure eigenvalues; to this end we factorize the polynomial $-\sigma^3 + C\sigma^2 + B\sigma + A$ in a quadratic polynomial of the form $\sigma^2 + \alpha_1^2$, so that

$$-\sigma^3 + C\sigma^2 + B\sigma + A = (\sigma^2 + \alpha_1^2)(-\sigma + \alpha_2) = -\sigma^3 + \alpha_2\sigma^2 - \alpha_1^2\sigma + \alpha_1^2\alpha_2.$$

To satisfy this condition, we must have $A = -BC$, i.e. $A + BC = 0$, in addition to the conditions $A, B, C < 0$. Moreover, a transversality condition must be hold. In view of the rather complicated coefficients these conditions are very difficult to investigate analytically. We therefore turn to numerical simulations. By varying the values of parameters, we obtain different situations.

In the Fig.1 the stable equilibrium behavior of the system is found by setting $\beta = 0.1$, $\gamma_1 = 0.5$, $\gamma_2 = 0.5$, $\lambda = 0.3$, $\mu = 0.1$, $\nu = 0.3$, $\pi = 0.3$, $\rho = 0.4$ and $m = 0.1$. Note that these choices for the parameter values satisfy the conditions $A, B, C < 0$ but not $A + BC = 0$.

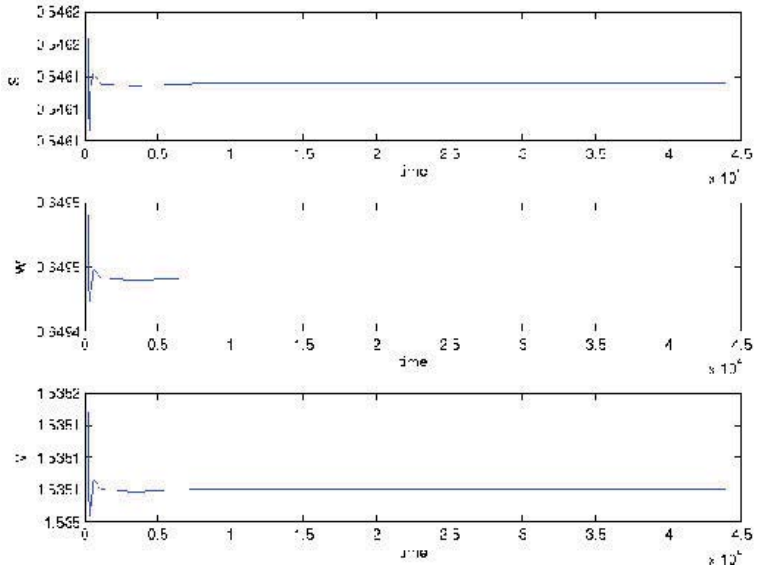

Figure 1: Here and in all subsequent figures, top to bottom, are respectively shown the populations S, W and V as functions of time. Stable coexistence equilibrium behavior for $\beta = 0.1$, $\gamma_1 = 0.5$, $\gamma_2 = 0.5$, $\lambda = 0.3$, $\mu = 0.1$, $\nu = 0.3$, $\pi = 0.3$, $\rho = 0.4$ and $m = 0.1$.

If we now try to change the parameter values, so that $A + BC$ is close to zero, we find the onset of oscillations, which stabilize as time flows, as shown in Fig.2, with $\beta = 0.1$, $\gamma_1 = 0.1$, $\gamma_2 = 0.9$, $\lambda = 0.1$, $\mu = 0.35$, $\nu = 0.01$, $\pi = 0.01$, $\rho = 0.05$ and $m = 0.1$.

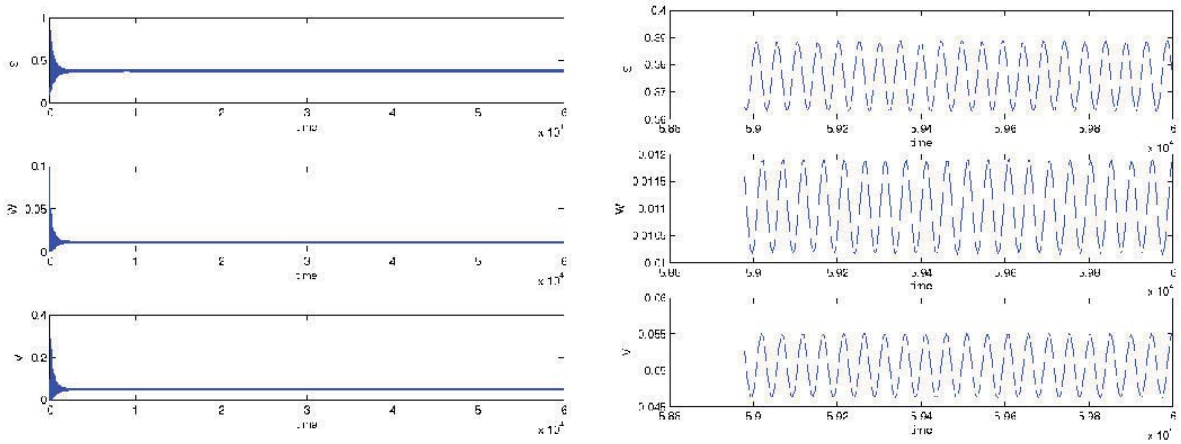

Figure 2: Onset of oscillations: $\beta = 0.1$, $\gamma_1 = 0.1$, $\gamma_2 = 0.9$, $\lambda = 0.1$, $\mu = 0.35$, $\nu = 0.01$, $\pi = 0.01$, $\rho = 0.05$ and $m = 0.1$. Left: the solutions in the time interval $[0, 60000]$; right: blow up of the solutions over the time interval $[58800, 60000]$.

Finally, for the same values of λ and γ_2, with the remaining parameter values given by $\beta = 0.003$, $\gamma_1 = 0.2$, $\mu = 0.06$, $\nu = 0.005$, $\pi = 0.1$, $\rho = 0.02$ and $m = 0.2$, we obtain the limit cycles behavior of Fig.3.

6 Special Cases

In this section we consider particular cases of (2).

Note that in the model β and γ_1 cannot vanish at the same time, as well as λ and γ_2. In such case indeed there would be no possibility of transitions into each of the infected classes.

6.1 No $W \to V$ transition

Assume that the weakly infected individuals cannot become strongly infected, i.e. $\lambda = 0$. The equilibria in addition to the origin are the points

$$E_1 = \left(\frac{\pi + m}{\beta}, \frac{\rho(\pi + m)}{m\beta}, 0 \right),$$

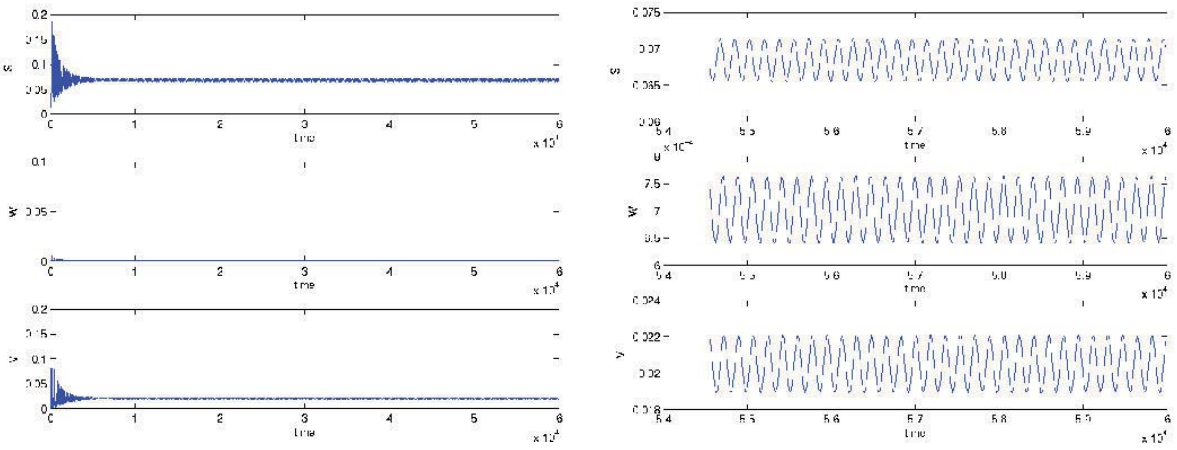

Figure 3: Limit cycles: $\beta = 0.003$, $\gamma_1 = 0.2$, $\gamma_2 = 0.9$, $\lambda = 0.1$, $\mu = 0.06$, $\nu = 0.005$, $\pi = 0.1$, $\rho = 0.02$ and $m = 0.2$. Left: the solutions in the time interval $[0, 60000]$; right: blow up of the solutions over the time interval $[54000, 60000]$.

which is feasible if and only if $\rho > 0$ and

$$E_2 = \left(\frac{\mu+\nu}{\gamma_2}, \frac{\gamma_1 \rho (\nu+\mu)^2}{\gamma_2(\gamma_1 m(\nu+\mu) - \mu(\beta(\nu+\mu) - \gamma_2(m+\pi)))}, \right.$$
$$\left. -\frac{(\nu+\mu)\rho(\beta(\nu+\mu) - \gamma_2(m+\pi))}{\gamma_2(\gamma_1 m(\nu+\mu) - \mu(\beta(\nu+\mu) - \gamma_2(m+\pi)))} \right).$$

For feasibility of the latter, $\rho > 0$ is necessary, together with the condition

$$\beta < \frac{\gamma_2(\pi+m)}{\nu+\mu}.$$

For E_0 the eigenvalues are $\sigma_1 = \rho$, $\sigma_2 = -\pi - m$, $\sigma_3 = -\mu - \nu$, so that the origin is stable if and only if $\rho < 0$. The characteristic equation at E_1 and at E_2 is a cubic of the form $-\sigma^3 + C\sigma^2 + B\sigma + A = 0$, with A, B e C functions of the parameters of the model.

6.2 V unrecoverable state, no $W \to V$ transition

We assume that the disease cannot progress from the mild form to the strong form and also that once an individual gets infected with the virulent form he cannot recover, i.e. $\nu = \lambda = 0$. The equilibria are again the origin $E_0 = (0,0,0)$ and the points

$$E_1 = \left(\frac{\pi+m}{\beta}, \frac{(\pi+m)\rho}{\beta m}, 0 \right), \tag{8}$$
$$E_2 = \left(\frac{\mu}{\gamma_2}, -\frac{\gamma_1 \mu \rho}{\gamma_2(-\gamma_1 m + \mu\beta - \pi\gamma_2 - m\gamma_2)}, \frac{(\mu\beta - \pi\gamma_2 - m\gamma_2)\rho}{\gamma_2(-\gamma_1 m + \mu\beta - \pi\gamma_2 - m\gamma_2)} \right)$$

E_1 is feasible if and only if $\rho > 0$, E_2 is feasible only for $\rho > 0$ and if

$$\beta < \frac{\gamma_2(\pi+m)}{\mu}.$$

The following are the eigenvalues at E_0: $\sigma_1 = \rho$, $\sigma_2 = -\pi - m$, $\sigma_3 = -\mu$. From this, the origin is stable if and only if $\rho < 0$.

The characteristic equations evaluated at E_1 and at E_2 are of the form

$$-\sigma^3 + C\sigma^2 + B\sigma + A = 0,$$

with coefficients for E_1 given by

$$A = \frac{1}{\beta}(\gamma_2\pi + \gamma_2 m - \mu\beta)\rho(\pi+m)$$
$$B = \frac{1}{m\beta}[(\gamma_2\pi + \gamma_2 m - \mu\beta)\rho\pi - \beta\rho m(\pi+m)]$$
$$C = \frac{1}{m\beta}[(\gamma_2\pi + \gamma_2 m - \mu\beta)m - \beta\rho\pi]$$

while letting $E = \gamma_2(\gamma_1 m - \mu\beta + \pi\gamma_2 + m\gamma_2) > 0$, those for E_2 are

$$\begin{aligned} A &= -\frac{1}{E}\left(\rho\mu\pi^2\gamma_2^2 + \rho\mu m^2\gamma_2\gamma_1 + \rho\mu\pi\gamma_2\gamma_1 m - 2\rho\mu^2\beta\pi\gamma_2\right. \\ &\quad \left. -2\rho\mu^2\beta m\gamma_2 + \rho\mu m^2\gamma_2^2 - \rho\mu^2\beta\gamma_1 m + \rho\mu^3\beta^2 + 2\rho\mu\pi\gamma_2^2 m\right) \\ B &= -\frac{1}{E}\left(-\rho\mu^2\gamma_2\beta + \rho\mu\gamma_2\gamma_1 m + \rho\mu\gamma_2^2 m + \rho\mu\gamma_2^2\pi + \gamma_1\rho m\mu\beta - \rho\mu^2\beta\gamma_1 + \rho\mu\pi\gamma_2\gamma_1\right) \\ C &= \frac{1}{E}\left(-2\pi\gamma_2^2 m + m^2\gamma_2\gamma_1 + m^2\gamma_2^2 + \mu^2\beta^2 + \pi^2\gamma_2^2 + \rho\gamma_1\pi\gamma_2\right. \\ &\quad \left. -\mu\beta\gamma_1 m + \pi\gamma_2\gamma_1 m - 2\mu\beta\pi\gamma_2 - 2\mu\beta m\gamma_2\right). \end{aligned}$$

Necessary and sufficient conditions for stability for both equilibria can be expressed by the following Routh-Hurwitz conditions

$$A < 0, \quad C < 0, \quad CB + A > 0,$$

thus implying also $B < 0$. Now $B < 0$ and $C < 0$ reduce to requiring respectively

$$\gamma_2(\pi + m) < \beta\left[\rho\mu\pi + m\rho(\pi + m)\right], \quad \gamma_2(\pi + m) < \beta\left(m\mu + \pi\rho\right). \tag{9}$$

These conditions show that one of them is redundant. In fact they reduce to just $B < 0$ or $C < 0$ according to the following inequality

$$m\mu + \rho\mu < \rho\mu\pi + m\rho(m + \pi).$$

Letting $\Gamma \equiv \gamma_2(\pi + m) - \beta\mu$, the condition $BC + A > 0$ can be rewritten as

$$\Gamma^2 + \beta^2\rho m(\pi + m) > \Gamma\beta\rho\pi. \tag{10}$$

An inspection of the coefficients for the case of E_1 shows that a large μ makes them and Γ negative, satisfying thus (9) and (10) as well. Therefore a large cumulative mortality rate for the strongly infected ensures the disappearance of the strong form of the disease.

Alternatively, to get easy conditions for instability, we can determine positive intersections of the two functions $B\sigma + A$ e $\sigma^3 - C\sigma^2$.

In order to have a Hopf-bifurcation, we need again to pure immaginary eigenvalues, and that entails to require $A + BC = 0$ in addition to the conditions $A, B, C < 0$. We investigate numerically this situation, as an example, we take $\lambda = \nu = 0$ and the other parameters with values given by $\beta = 0.003$, $\gamma_1 = 0.2$, $\gamma_2 = 0.9$, $\mu = 0.06$, $\pi = 0.1$, $\rho = 0.02$ and $m = 0.2$, see Fig.4. Under this regime, we find persistent oscillations.

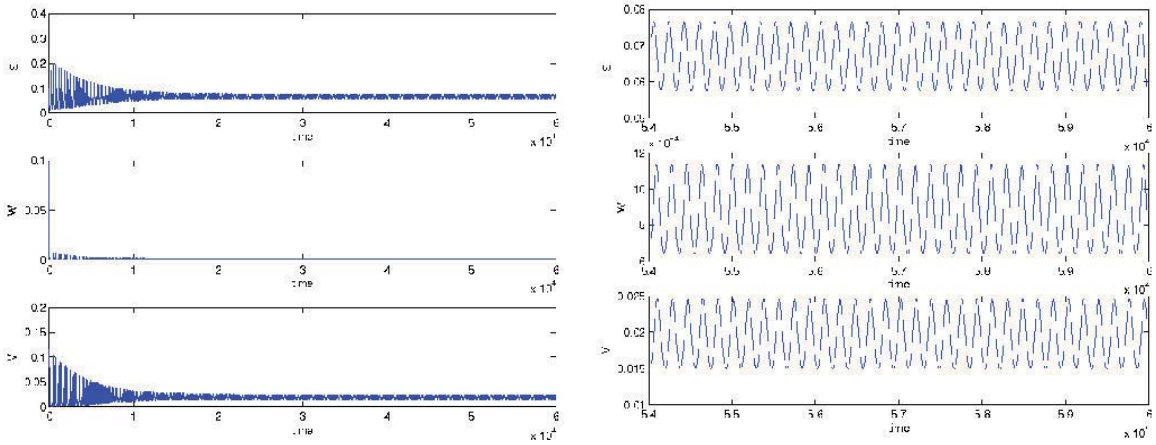

Figure 4: $\lambda = \nu = 0$, $\beta = 0.003$, $\gamma_1 = 0.2$, $\gamma_2 = 0.9$, $\mu = 0.06$, $\pi = 0.1$, $\rho = 0.02$ and $m = 0.2$. Left: the solutions over $[0, 60000]$; Right: blow up of the solutions over $[54000, 60000]$.

6.3 Weak form of the disease is not contagious

We assume here that susceptibles are unable to contract the disease via contacts with the weakly infected, i.e. $\beta = 0$.

The equilibria in this case are again the origin, and the point

$$E_1 = \left(\frac{(\lambda + m\pi)(\nu + \mu)}{(\lambda + m + \pi)\gamma_2 + \lambda\gamma_1}, \frac{\rho\gamma_1(\lambda + m\pi)(\nu + \mu)^2}{((\lambda + m + \pi)\gamma_2 + \lambda\gamma_1)(\mu((\lambda + m + \pi)\gamma_2 + \lambda\gamma_1) + m\gamma_1(\nu + \mu))}, \right.$$
$$\left. \frac{\rho(\lambda + m\pi)(\nu + \mu)}{\mu((\lambda + m + \pi)\gamma_2 + \lambda\gamma_1) + m\gamma_1(\nu + \mu)}\right),$$

which is feasible if and only if $\rho > 0$.

At E_0 the eigenvalues are $\sigma_1 = \rho$, $\sigma_2 = -\lambda - \pi - m$, $\sigma_3 = -\mu - \nu$ so that the origin is stable if and only if $\rho < 0$. In a similar way as already seen for the other cases, the characteristic equation for E_1 is a cubic of the form $-\sigma^3 + C\sigma^2 + B\sigma + A = 0$, with A, B e C depending on parameters of the model in a complicated way.

6.4 Virulent contagion leads always to severe cases

Assume now that susceptibles cannot be weakly infected via contacts with strongly infected, i.e. $\gamma_1 = 0$. There are in this case two possible equilibria, other than the origin, namely

$$E_1 = \left(\frac{\nu + \mu}{\gamma_2}, 0, \frac{(\nu + \mu)\rho}{\gamma_2 \mu} \right),$$

which is feasible if and only if $\rho > 0$, and

$$E_2 = \left(\frac{\lambda + \pi + m}{\beta}, \frac{(\beta(\nu + \mu) - \gamma_2(\lambda + \pi + m))\rho(\lambda + \pi + m)}{\beta(\lambda\mu\beta + m(\beta(\nu + \mu) - \gamma_2(\lambda + \pi + m)))}, \frac{\rho\lambda(\lambda + \pi + m)}{\lambda\mu\beta + m(\beta(\nu + \mu) - \gamma_2(\lambda + \pi + m))} \right).$$

For feasibility of the latter, $\rho > 0$ is required, together with the additional condition

$$\beta > \frac{\gamma_2(\lambda + \pi + m)}{\nu + \mu}.$$

Once again, the origin has the eigenvalues $\sigma_1 = \rho$, $\sigma_2 = -\lambda - \pi - m$, $\sigma_3 = -\mu - \nu$ and therefore it it is stable if and only if $\rho < 0$ and for E_1, E_2 we obtain again a cubic characteristic equation similar to the one seen above for the other cases.

6.5 Severe stage cannot be obtained by contact with V's

Here we assume that susceptibles cannot contract the strong form of the disease via direct contacts with the strongly infected, i.e. $\gamma_2 = 0$. The equilibria are the origin and

$$E_1 = \left(\frac{(\lambda + \pi + m)(\nu + \mu)}{\beta(\nu + \mu) + \gamma_1\lambda}, \frac{\rho(\lambda + \pi + m)(\nu + \mu)}{(\beta(\nu + \mu) + \gamma_1\lambda)(m(\nu + \mu) + \lambda\mu)(\nu + \mu)}, \right.$$
$$\left. \frac{\rho\lambda(\lambda + \pi + m)(\nu + \mu)}{(\beta(\nu + \mu) + \gamma_1\lambda)(m(\nu + \mu) + \lambda\mu)} \right),$$

which is feasible if and only if $\rho > 0$.

E_0 has the following eigenvalues $\sigma_1 = \rho$, $\sigma_2 = -\lambda - \pi - m$, $\sigma_3 = -\mu - \pi$, so that the origin is stable if and only if $\rho < 0$. Also for E_1 we obtain a cubic equation as in the former cases.

7 Numerical investigation of the limit cycles

In this section we analyze what happens to the oscillations earlier discovered in the model, focusing on the influence each system parameter has on their properties. We begin by proposing a reference figure, for some values similar to those of Figure 3, this time showing it in a smaller time interval, and at the same time explicitly plotting also the prevalence of each subpopulation, i.e. the ratios of S, W and V over the whole population N, in the diagrams on the right, see Figure 5.

Figures 6-13 contain the changes occurring by varying in turn each parameter. More specifically, we find that for a high enough value of $\beta = 3.7$ the oscillations damp down to the equilibrium value E_1, showing that this can be taken as bifurcation parameter, Figure 6.

Figure 7 shows instead that the limit cycles are essentially indifferent to changes in the parameters γ_1 and γ_2. The main changes concern the length of each oscillation, where longer periods of low infectivity are followed by epidemics outbreaks leading to much higher prevalences in the class of strongly infected, while the class of weakly infected seems to be less affected.

Similar considerations hold for changes in the parameter λ, Figure 8. The lower value $\lambda = 0.001$ shows higher frequecies in the oscillations, the higher value $\lambda = 0.8$ leads to longer periods of very low endemicity, followed by short epidemics outbreaks, with heavy prevalence for the virulent form of the disease.

The same behavior is observed in the changes for the parameter m, in the range $[0.002, 2.2]$. As it becomes larger, limit cycles periods become increase while their amplitudes also increase for the S and V classes, while decreasing for the W class, Figure 9.

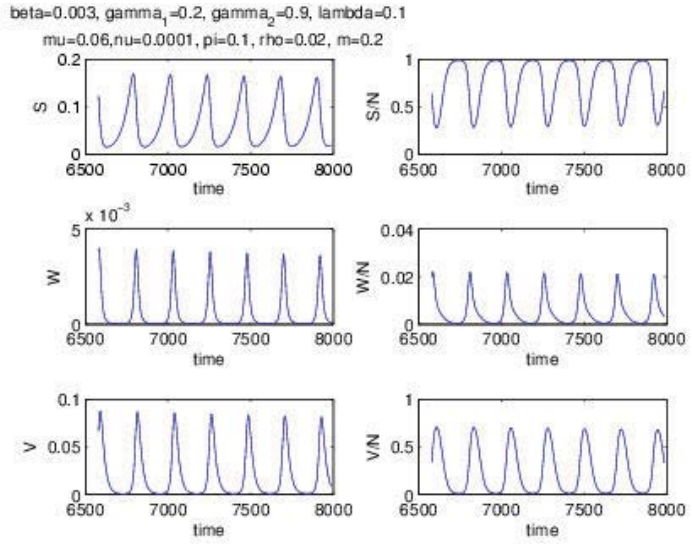

Figure 5: Reference figures for the parameter values $\beta = 0.003$, $\gamma_1 = 0.2$, $\gamma_2 = 0.9$, $\lambda = 0.1$, $\mu = 0.06$, $\nu = 0.0001$, $\pi = 0.1$, $\rho = 0.02$ and $m = 0.2$. On the left the subpopulations, on the right the prevalence of each class, all as functions of time.

In Figure 10 we observe that the changes in frequency and periods of the oscillations is much more evident, due to the higher sensitivity of the system with respect to the parameter μ, at low values, $\mu = 0.003$ showing longer periods, and at higher values $\mu = 0.57$ much higher frequencies than ever noticed in other previous parameter changes.

Decreasing the value of the parameter ν to zero, or to values close to it, gives again longer periods of oscillations, while for the higher value $\nu = 0.045$ we recover once more the stable equilibrium E_1, Figure 11.

Changing π to low values gives again longer periods, $\pi = 0.005$ and to the larger value $\pi = 2.4$ shows higher frequencies, Figure 12.

Finally, setting $\rho = 0.004$ gives the longest periods of all the simulations, and letting $\rho = 3.8$ recovers once again the stable equilibrium point E_1, Figure 13.

8 Some inferences

We proposed a model for a disease with a mild and a virulent stage raging in a population which reproduces. Our results show that if the net birth rate of the population is negative, the system collapses and the population is wiped out. However if the net birth rate is positive, there occur two possible equilibria: the origin now becomes unstable and an interior endemic equilibrium now arises. We have shown numerically that it bifurcates to originate stable limit cycles around it, for suitable choices of the parameter values.

An interpretation of the stable oscillations in all the subpopulations indicates that they correspond to reiterated epidemics outbreaks. Therefore it may be much better to avoid them and rather keep the disease at a low level, though endemic, than having the cycles which may lead to alternating low and very high peaks in the infected populations.

The results of the simulations carried out in the previous Section indicate that some parameters are less relevant to assess the dynamics of the epidemics. Among these, we certainly find γ_1 and γ_2, which influence mainly the periods of the cycles and their amplitudes. If the time between successive disease outbreaks is of no too much concern, also λ, m, μ and π are not too relevant. Thus the strong disease incidences γ_1 and γ_2 do not seem to play a relevant role, in this situation, nor the transition rate from the class of weakly infected to the virulent stage of the disease. The natural mortality also is of small relevance, as in a sense it could be intuitive in an epidemics model. Also surprisingly the recovery rates from the mild stage of the disease into the susceptibles plays a marginal role here. All these parameters become thus relevant only if one wants to delay the epidemics outbreaks and space them more in time. Measures taken to act on these parameters so that their values change in the directions indicated by the simulations will then be beneficial toward this goal. For the natural plus disease related mortality of the severely infected class, μ, this is particularly evident.

Instead β, μ and ρ control the onset of the limit cycles. Thus a high contact rate with the weak form of the infected leads to the stable equilibrium. It is when this incidence drops below a threshold that cycles start to appear. The recovery rate from severely infected directly to susceptibles instead has a relevant role in setting the

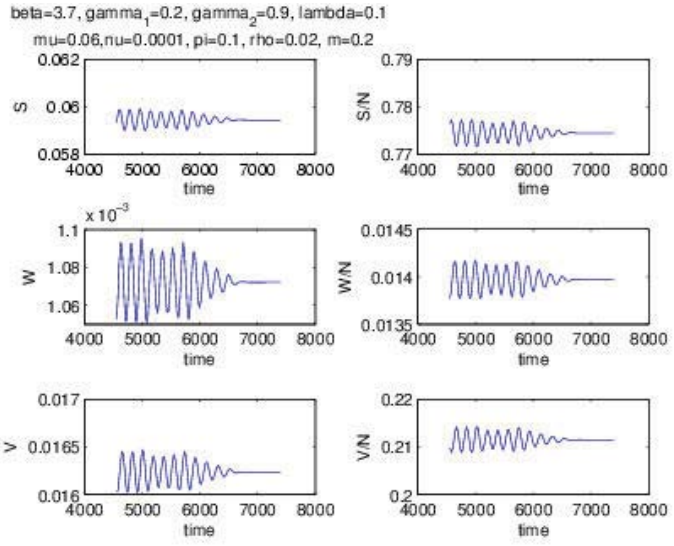

Figure 6: Parameter changed is here $\beta = 3.7$. On the left the subpopulations, on the right the prevalence of each class, all as functions of time.

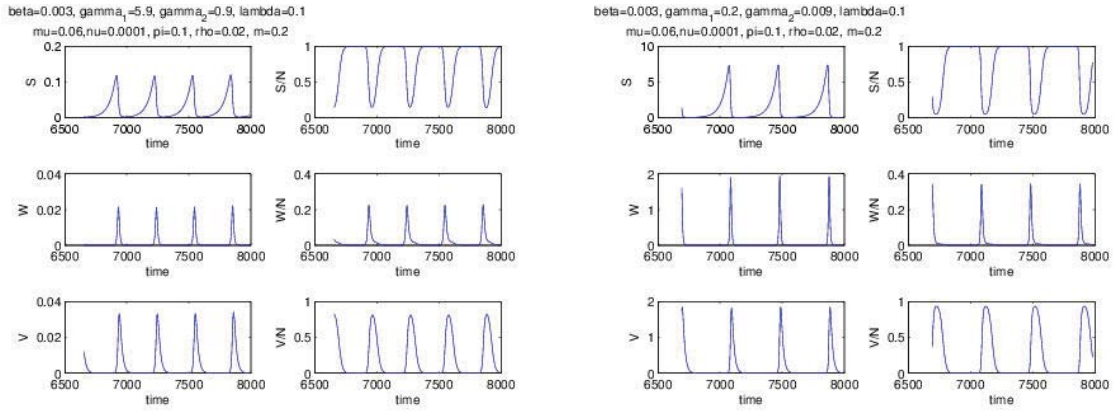

Figure 7: Parameter changed are here $\gamma_1 = 5.9$ (left) and $\gamma_2 = 0.009$ (right). On the left the subpopulations, on the right the prevalence of each class, all as functions of time.

disease to an endemic form, if it has a high enough value. The net birth rate of susceptibles instead appears to be the most important parameter in this model. In fact for very low values, one is able to spread the epidemics in time, having them to appear only after long time intervals. For high enough values instead, the dynamics of the disease settles toward the equilibrium point E_1. In this case Figure 13 shows a very high prevalence of strongly infected individuals, but by suitably influencing the other parameters it is perhaps possible to obtain a lower such prevalence. This is shown in Figure 14 for the parameter values $\rho = 3.8$, $\nu = 0.2$ and $\beta = 2.3$, where the prevalences of both the mild and virulent forms of the disease become much smaller.

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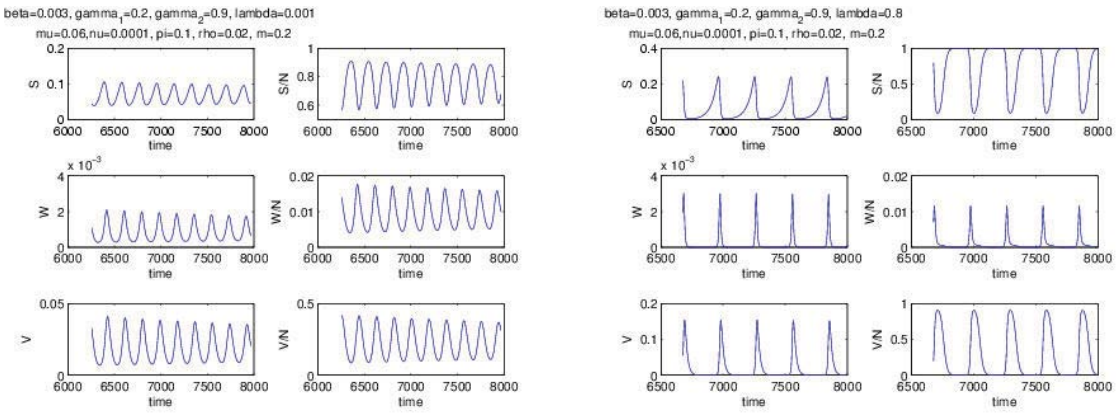

Figure 8: Parameter changed is here $\lambda = 0.001$ (left) and $\lambda = 0.8$ (right). On the left the subpopulations, on the right the prevalence of each class, all as functions of time.

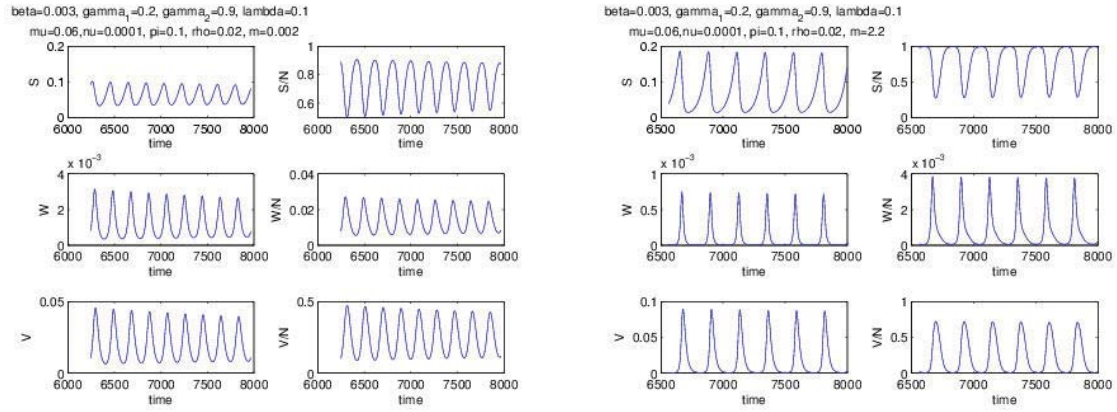

Figure 9: Parameter changed is here $m = 0.002$ (left) and $m = 2.2$ (right). On the left the subpopulations, on the right the prevalence of each class, all as functions of time.

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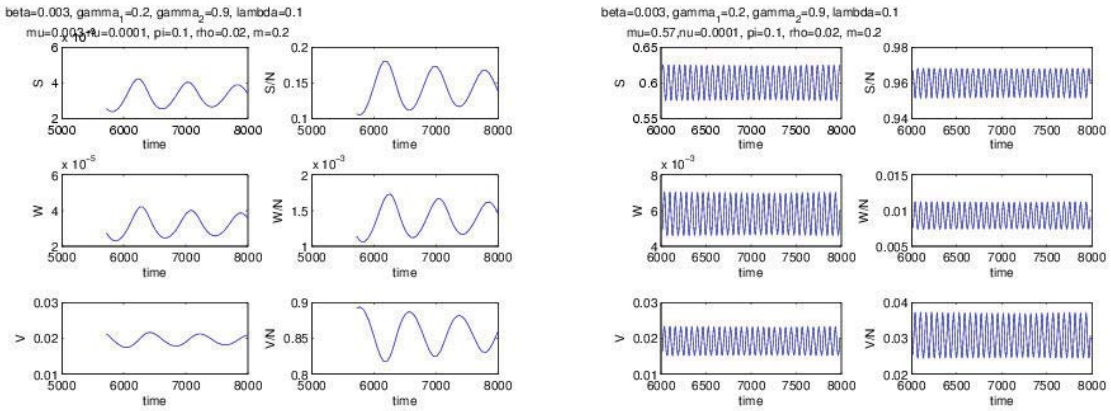

Figure 10: Parameter changed is here $\mu = 0.003$ (left) and $\mu = 0.57$ (right). On the left the subpopulations, on the right the prevalence of each class, all as functions of time.

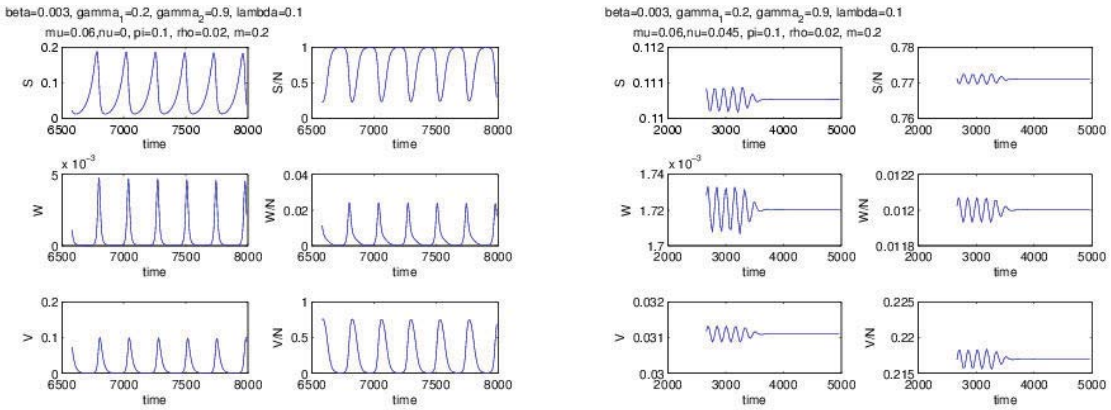

Figure 11: Parameter changed is here $\nu = 0.0$ (left) and $\nu = 0.045$ (right). On the left the subpopulations, on the right the prevalence of each class, all as functions of time.

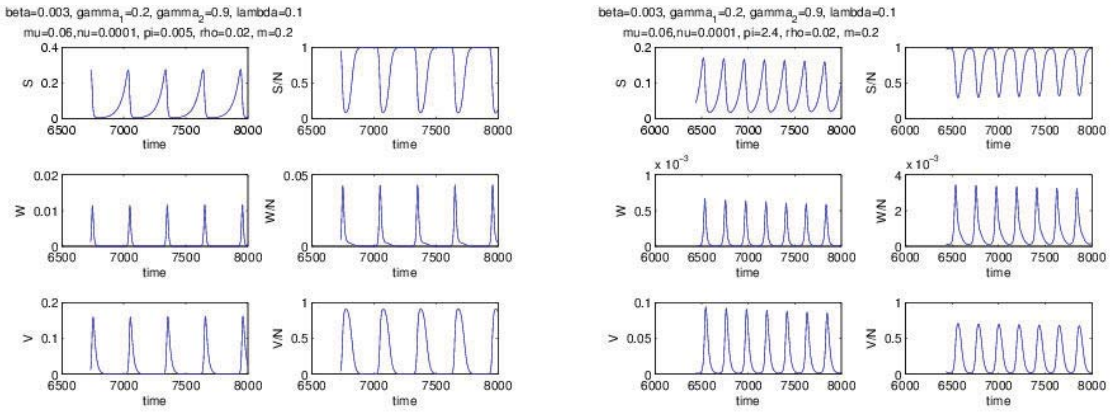

Figure 12: Parameter changed is here $\pi = 0.005$ (left) and $\pi = 2.4$ (right). On the left the subpopulations, on the right the prevalence of each class, all as functions of time.

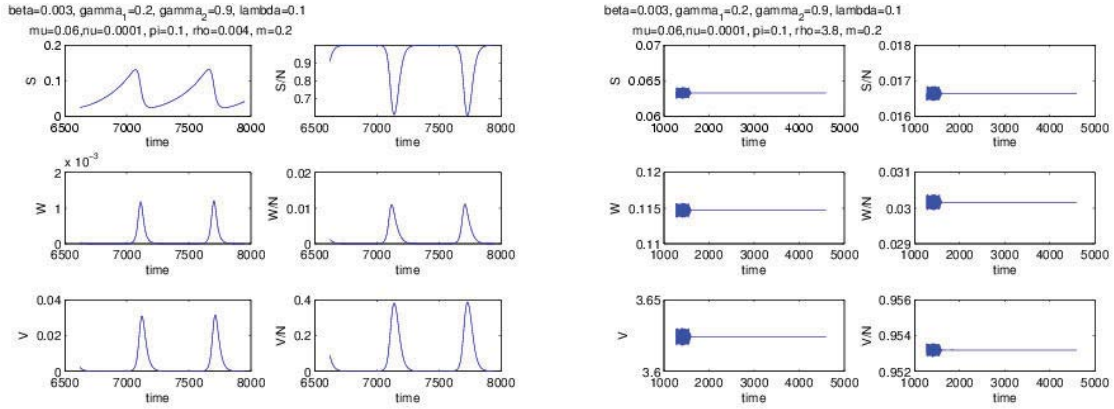

Figure 13: Parameter changed is here $\rho = 0.004$ (left) and $\rho = 3.8$ (right). On the left the subpopulations, on the right the prevalence of each class, all as functions of time.

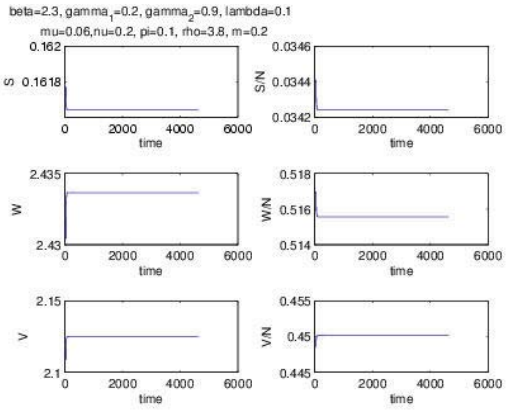

Figure 14: Control of the disease prevalences by acting on the parameters $\nu = 0.2$ and $\beta = 2.3$ while keeping $\rho = 3.8$, compare with Figure 13. On the left the subpopulations, on the right the prevalence of each class, all as functions of time.