TAYLORIAN INITIAL PROBLEM

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Abstract. In recent years, intensive research in the field of numerical solutions of systems of ordinary and partial differential equations has been done at the Brno University of Technology, Faculty of Information Technology, Department of Intelligent Systems. The basic numerical method employed is the so-called Modern Taylor Series Method (MTSM). It has been described, studied, and numerous aspects have been investigated such as processing in parallel systems. Also a simulation system TKSL has been developed which is based on the Taylor series method. For some results see [5], [1]. Although there have been considerable practical results, theoretical issues are yet to be investigated. A theoretical background of the method, some succesful results, some comparisons to word standards and idea of parallel processing will be provided in this paper.

1 Introduction

The MTSM is based on a transformation of the initial problem into another initial problem with polynomials on the right-hand sides. This is a precondition for a Taylor series method to be successfully applied to the task of finding a numerical solution. The solution of the transformed initial problem then includes the solution of the original system.

A transformation similar to that described in this paper can be found in [2]. Practical aspects are discussed in [4], and [5]. An outline of the special type transformations for functions commonly encountered in initial problems can be found in [3].

As an example, a transformation into polynomial form (into Taylorian initial problem) the following initial problem

$$x_1' = \frac{1}{\sin x_1}, \ x_1(0) = \frac{\pi}{2} \tag{1}$$

is presented. Putting

$$\frac{1}{\sin x_1} = x_2$$

we can construct

$$\begin{aligned} x_1' &= x_2 & x_1(0) &= \frac{\pi}{2} \\ x_2' &= -x_2^3 \cos(x_1) & x_2(0) &= 1 \end{aligned} \tag{2}$$

Putting $x_3 = \cos(x_1)$, $x_4 = \sin(x_1)$ we obtain a Taylorian initial problem

$$\begin{aligned} x_1' &= x_2 & x_1(0) &= \frac{\pi}{2} \\ x_2' &= -x_2^3 \cdot x_3 & x_2(0) &= 1 \\ x_3' &= -x_2 \cdot x_4 & x_3(0) &= 0 \\ x_4' &= x_2 \cdot x_3 & x_4(0) &= 1 \end{aligned} \tag{3}$$

We can see that all the expressions on the right-hand sides of Eq. (3) are polynomials.

The idea above requires software capable of automatically performing the decomposition of the right-hand sides of ordinary differential equations.

This approach has been implemented in a simulation language TKSL (an implementation of the Modern Taylor Series Method on a personal computer).

In fact, the well-known rules of differential and integral calculus have been used.

2 Automatic integration method order setting

The best-known and most accurate method of calculating a new value of a numerical solution of a differential equation is to construct the Taylor series in the form

$$y_{n+1} = y_n + h * f(t_n, y_n) + \frac{h^2}{2!} * f^{[1]}(t_n, y_n) + \cdots + \frac{h^p}{p!} * f^{[p-1]}(t_n, y_n) \tag{4}$$

where h is the integration step.

The main idea behind the Modern Taylor Series Method is an automatic integration method order setting, i.e. using as many Taylor series terms for computing as needed to achieve the required accuracy.

The Modern Taylor Series Method used in the computations increases the method order ORD automatically, i.e. the values of the terms

$$\frac{h^p}{p!} * f^{[p-1]}(t_n, y_n)$$

are computed for increasing integer values of p until adding the next term does not improve the accuracy of the solution.

2.1 Van Der Pol's equation

As an application of the automatic integration method order setting the solution of the well known Van-Der-Pol's equation in the form of a system first order differential equations

$$y' = y_1 \qquad y(0) = 0 \tag{5}$$

$$y_1' = -y - 3 \cdot (y^2 - 1) y_1 \qquad y_1(0) = 2 \tag{6}$$

is described.

Time functions ORD (ORD stands for the method order), y and y_1 are in Figure 1. The aim of Figure 1 is to point out two things. First, the values of ORD are high and second, these values vary considerably during the computation.

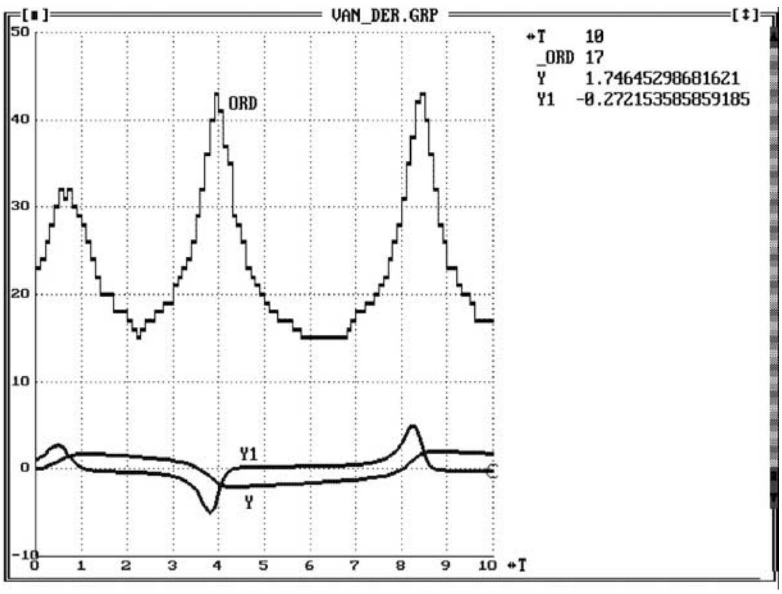

Figure 1: Demonstration of a Van Der Pool equation

2.2 Stiff systems

Another application of the automatic integration method order setting is focused on a particular problems with the integration of stiff systems. Let us consider a system of linear differential equations

$$y_1' = y_2 \qquad y_1(0) = 1 \tag{7}$$

$$y_2' = -a \cdot y_1 - (a+1) \cdot y_2 \qquad y_2(0) = -1 \tag{8}$$

Figure 2 illustrates problems of stiff systems. The part labelled STIFEX1 shows the solution and the function ORD for the system (7),(8)). This is a case when the difference of eigenvalues is small (a=100).

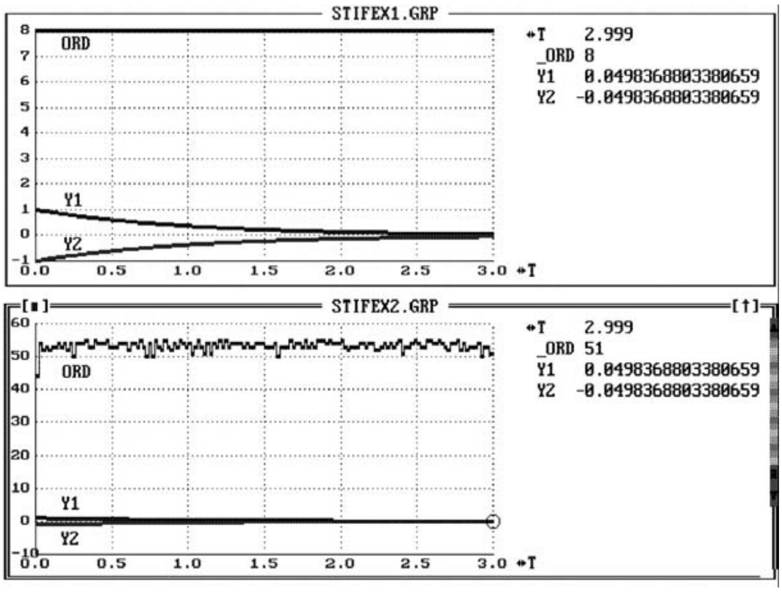

Figure 2: Demonstration of stiff systems

If the difference of eigenvalues of the system is large (a=1000) (system (7),(8)), there is a significant increase in the value of ORD (in the part labelled STIFEX2).

2.3 Polynomial functions

If the right-hand side of a first order differential equation (9) has the form of a polynomial of a degree k, then the integration method of order = k+1 will ensure the absolute accuracy of calculations with an arbitrary integration step.

$$y' = a \cdot t^k \qquad y(0) = y_0 \tag{9}$$

The numerical solution of Eq. (9) by the Taylor series method is in Eq. (10)

$$y_{n+1} = y_n + h \cdot a \cdot t_n^k + \frac{h^2}{2} \cdot a \cdot k \cdot t_n^{k-1} + \cdots + \frac{h^{k+1}}{k+1} \cdot a \tag{10}$$

The integration step h can be chosen arbitrarily; the numerical solution of (9) will be absolutely accurate.

Note: *If the order of the integration method is less than $k+1$, we can get the numerical solution of the differential equation (9) only approximately. The accuracy, in such a case, is dependent on the integration step.*

2.4 Definite Integrals

Definite integrals and integral equations, due to the number of applications, are very important mathematical tools. Their solution using the Modern Taylor Series Method is also the subject of this paper.

For all functions that have Taylor series the calculation of their integrals can be conducted indirectly via their derivatives. Thus, the problem of solving a definite one-dimensional integral taken as a function of the upper boundary can be transformed to solving a system of differential equations.

Let a definite integral (11) be given.

$$y = \int_0^{\pi} \sin(t) dt \tag{11}$$

The definite integral can be rewritten in the form (12):

$$Y' = Z \tag{12}$$
$$Z = \sin(t) \tag{13}$$

The initial conditions can be obtained by substituting the value of the lower boundary for the variable t of the function Z.

The numerical solution of the integral is obtained at the point corresponding to the upper boundary of the integral ($t_{max} = \pi$).

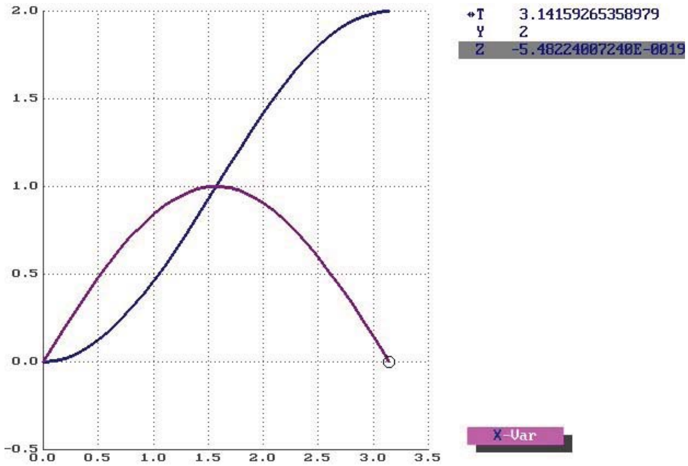

Figure 3: Definite integral computed using TKSL/386

The time functions of the value of the integral y and the function z being integrated are in Figure 3. A particular calculation of the integral for $T = \pi$ is in the right-hand part of Figure 3.

2.5 Fourier Analysis

A function $y(t)$ which is periodic of period 2π, is continuous except for a finite number of finite discontinuities in any interval of length 2π, and has a right-hand and left-hand derivative at every point may be expanded into a Fourier series Eq. (14)

$$y(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kt) + \sum_{k=1}^{\infty} b_k \sin(kt) \tag{14}$$

where the Fourier coefficients are given by Eq. (15), (16)

$$a_k = \frac{1}{\pi} \int_0^{2\pi} y(t)\cos(kt)dt \tag{15}$$

$$b_k = \frac{1}{\pi} \int_0^{2\pi} y(t)\sin(kt)dt \tag{16}$$

The definite integrals can be rewritten according to the illustrative in the form (17), (18)

$$a'_k = \frac{1}{\pi} y(t)\cos(kt) \tag{17}$$

$$b'_k = \frac{1}{\pi} y(t)\sin(kt) \tag{18}$$

The numerical solution of the integral is again obtained at the point corresponding to the upper boundary of the integral ($t_{max} = 2\pi$).

2.6 Exponential functions

Let the following differential equations be

$$y' = b{\cdot}y{\cdot}\cos(t) + y{\cdot}\sin(t){\cdot}(3{\cdot}a{\cdot}\sin(2{\cdot}t) - b{\cdot}t) \qquad y(0) = 1 \tag{19}$$

$$x' = -b{\cdot}x{\cdot}\cos(t) - x{\cdot}\sin(t){\cdot}(3{\cdot}a{\cdot}\sin(2{\cdot}t) - b{\cdot}t) \qquad x(0) = 1 \tag{20}$$

and the test function is

$$z = x \cdot y \tag{21}$$

Analytical solution of y Eq. (19) is known and is in the form

$$y = e^{(b \cdot t cos(t) + 2 \cdot a \cdot \sin^3(t))} \tag{22}$$

similarly analytic solution of Eq. (20) is

$$x = e^{(-(b \cdot t cos(t) + 2 \cdot a \cdot \sin^3(t)))} \tag{23}$$

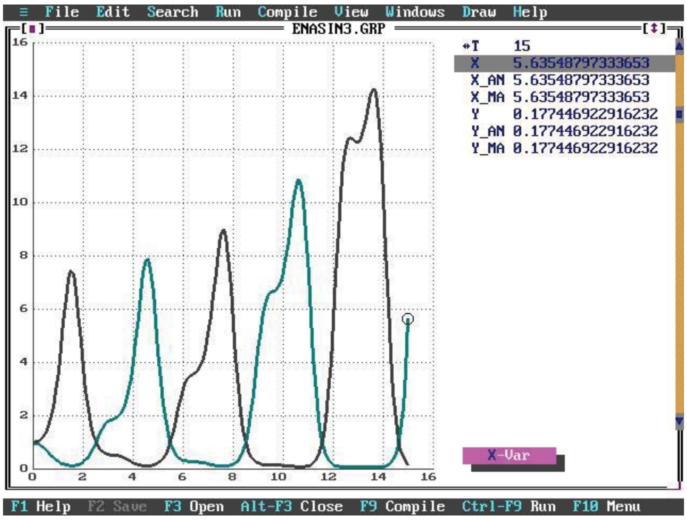

Figure 4: TKSL solution $a = 1$, $b = 0.2$

Analytic solution of the Eq. (19) can be obtained using Maple software in the form

$$y = e^{(\frac{3}{2} \cdot a \cdot \sin(t) + \cos(t) \cdot b \cdot t - \frac{1}{2} \cdot a \cdot \sin(3 \cdot t))} \tag{24}$$

$$x = e^{(-\frac{3}{2} \cdot a \cdot \sin(t) - \cos(t) \cdot b \cdot t + \frac{1}{2} \cdot a \cdot \sin(3 \cdot t))} \tag{25}$$

Actually seven time functions are in Figure 4 - Y, X, Z are the numerical solutions obtained directly by TKSL, Y_AN, X_AN are the analytical solutions Eq. (22), (23) and Y_MA, X_MA are the analytical solutions obtained by Maple Eq. (24), (25).

A quality of solution of system of Eq. (19), (20) can be checked using test function z (Eq. (21)). For the test function $z = x \cdot y$ we have $z = 1$ since the exact solution of Eq. (19) is $y = e^{(b \cdot t cos(t) + 2 \cdot a \cdot \sin^3(t))}$ and the exact solution of Eq. (20) is $x = e^{(-(b \cdot t cos(t) + 2 \cdot a \cdot \sin^3(t)))}$. Result of test function z for $a = 100$ in TKSL can be seen in Figure 5 ($z = 1$ even for final time $t = 145,68$s).

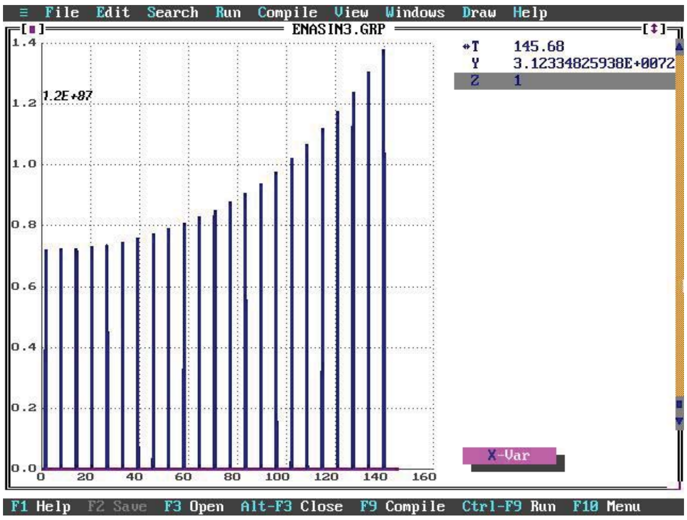

Figure 5: TKSL solution $a = 100$, $b = 0.2$

Note: *The graph of test function Eq. (21) computed by MatLab software can be seen in Figure 6 (solution z is plotted only to final time $t = 45s$; $z(45) = 1,91$ instead of $z = 1$).*

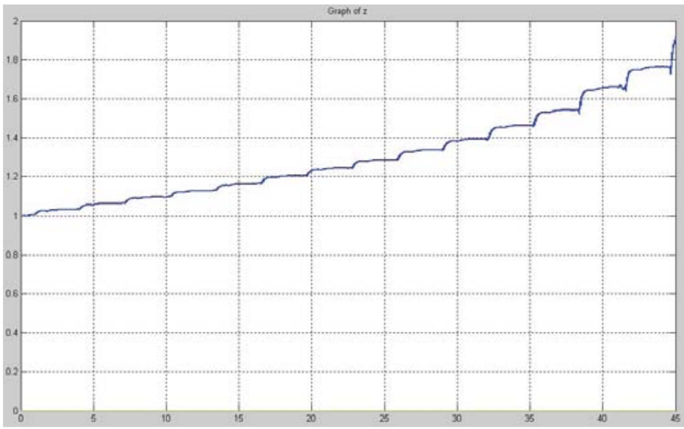

Figure 6: MATLAB ode45 solver z $a = 10$, $b = 0.2$

3 Conclusions

The principal objective of the paper is to provide an idea of a mathematical background for the practical research in the field of numerical solutions of systems of differential equations that has been conducted at the Brno University of Technology Faculty of Information Technology, Department of Intelligent Systems for some time. This particularly concerns the problem of transforming a system of differential equations into a new system with polynomials on the right-hand sides.

Another problem that the paper deals with has also arisen in practical research. It is the problem of a suitable stopping rule, that is a rule that decides that no other step needs to be undertaken since the set accuracy requirements are satisfied. Various methods can be devised, more or less ingenious, but the basic question remains, namely, at a given point in calculation, how many subsequent steps must actually be "inspected" to get a satisfactory conviction that the sum of increments provided by the rest of the infinite series will not exceed the required accuracy. In particular, when solving some relatively simple problems using the basic version of the Taylor series method algorithm, situations occurred where the calculation could not be stopped using a straightforward stopping rule since all the contributions equalled zero for several consecutive terms. Knowing the exact solutions to these particular problems it was clear that, in extremely unfavourable circumstances, the calculation might end with considerable errors. This can be helped by using an improved version of the basic Taylor series method algorithm.

Analysing the time function ORD is of great importance - especially when stiff systems are solved.

4 References

[1] Web sites, High performance computing. `http://www.itsolutions.cz/TKSL`.

[2] D. Barton. Taylor series and stiff equations. *ACM Transactions on Mathematical Software*, 6(3):280–294, 1980.

[3] D. Barton, M. Willers, I., and M Zahar, V. Taylor series methods for ordinary differential equations-an evaluation. *Mathematical Software*, pages 369–389, 1972.

[4] F. Chang, Y. and G. Corliss. Solving ordinary differential equations using taylor series. *ACM Transactions on Mathematical Software*, 8(2):114–144, 1982.

[5] J. Kunovský. *Modern Taylor Series Method*. FEI-VUT Brno, 1994. Habilitation work.