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MODELLING, SIMULATION AND CONTROL OF A REDUNDANT PARALLEL ROBOT USING INVARIANT MANIFOLDS

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Abstract. The aim of this contribution is a systematic approach of modelling, simulation and control of a parallel robotic manipulator. Regarding the framework of structured analysis of dynamical systems, the mathematical description yields a set of differential equations and some additional holonomic constraints which appear as algebraic relations between single state variables. By defining a number of ficticious additional input and output variables this descriptor representation can be interpreted as coupling or constraint control problem. Numerical stable simulation and exterior control for trajectory tracking can now be achieved by designing an apropriate state feedback that keeps the closed loop system on an invariant manifold of consistent/allowed states.

1 Introduction

In recent years the interest in parallel robotic manipulators grew steadily. This is due to several advantages based on the specific architecture of this kind of mechanical system. First of all since the electric drives are mounted on a fixed platform they do not have to be moved with the links of the robot which certainly lowers inertia. As a further consequence also the links and therefore the whole structure may be chosen significantly lighter. This clearly enables larger accelerations and shorter cycle times, which is of great importance for industrial applications. Another advantage is the fact that due to the multiple kinematic loops such parallel robots may be prestressed by the help of antagonistic torques which can be chosen independently of the aspired trajectories. Therefore backlash may be reduced to a minimum in order to increase precision.

In the past the process of mathematical modelling has been subject to a bunch of examinations which can be found in [1] and [4] for instance. Compared to classical industrial robots having a tree structure there are major problems occuring typically during the process of modelling such parallel robotic devices. In contrast to classical industrial robots that are well understood and consist of only one open serial kinematic chain these parallel robots with several closed chains are considerably harder to describe. Depending on the nature of the holonomic constraints, which are typically appearing in this context, in many cases even the number of the mechanical degrees of freedom and therefore a suitable set of generalized coordinates is difficult to find or unknown. Often the closed chains are cut open at some joints such that the system is getting a tree structure. In the following auxiliary dependent coordinates are introduced to simplify the process of modelling and to describe the single branches. At this point two different approaches can be followed as far as the literature is concerned. Using the given holonomic constraints of the cut joints these dependent coordinates may be eliminated to obtain the equations of motion involving a minimal set of coordinates. Difficulties arise from the fact that this elimination requires the analysis of the possibly ambiguous inverse kinematics which often even leads to non unique expressions since the coordinates contribute to the kinematic equations in a nonlinear fashion. In addition the resulting equations of motions are unhandy such that the numerical integration is quite involved. In the second approach that can be found frequently the time derivative of the holonomic constraints is used rather than the constrained equations themselves. Since this time derivative is linear with respect to the generalized velocities the latter which are not independent can be eliminated easily from the equations of motion. This reduces the number of equations of motion to a minimum. Unfortunately the whole set of kinematic equations has to be kept in the model for numerical integration. Therefore the number of ordinary differential equations which have to be integrated is not minimal. Because of the fact that the holonomic constraints are interpreted as nonholonomic even though they are integrable the dynamical system is embedded into a higher dimensional state space. The resulting differential equations are often computationally less demanding. Otherwise it has to be ensured that the initial condition for the numerical analysis belongs to the space of allowed configurations which may be a non trivial task for some involved kinematic relations. Furthermore any error due to numerical precision will result in states that are not physically motivated and without any additional effort there is no mechanism to bring the system back onto the manifold of admissible states.

The aim of this contribution is a systematic approach of modelling, simulation and control of parallel robotic manipulators that gets along without the described elimination of auxiliary coordinates or the definition of some integrable nonholonomic constraints. The planar redundant robot shown in figure 1 serves as an example. Besides the system simulation the goal will be to let the tool center point (TCP) follow desired trajectories. The TCP is connected to the ground by three symmetric two-link arms consisting of one passive and one active link each. The

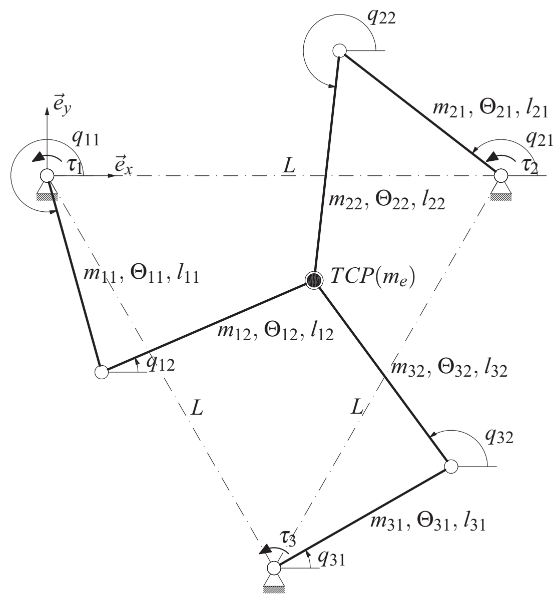

Figure 1: Planar redundant parallel robot

three drives at the active joints are represented by three independent torques τ_1, τ_2, τ_3.

2 Mathematical modelling

Regarding the framework of structured analysis of dynamical systems, it is often useful and convenient to divide the overall system into smaller parts. This way of modelling is quite intuitive and less complex, even though the aggregation of those subsystems requires some additional compatibility conditions. Considering the standard state space approach, these conditions appear as algebraic equations defining relations between single state variables. Such differential-algebraic systems are also known as descriptor systems.

As in classical approaches first of all the robotic system will be cut into three subsystems, as shown in the free-body diagram depicted in figure 2. In contrast to the well known procedures a state space representation in descriptor form will be derived including the required holonomic constraints directly. Subsequently the formulation of a appropriate control problem will be addressed to the obtained differential-algebraic system, that enables the controller design and system simulation with known nonlinear design methods and standard integration algorithms.

2.1 Lagrange's formalism

The derivation of the equations of motion of the three robotic arms as subsystems is a standard problem in multibody dynamics. Either Kane's method or energy-based methods as the well-known Hamilton or Lagrange equations may be used (see [3], [6] and [9]). In this paper because of the simple kinematic relations of the subsystems the latter approach will be followed.

For the i-th arm with $i = 1,2,3$ the centres of mass of the two rigid bodies and the TCP are given by the kinematic equations

$$\begin{aligned}
\vec{r}_{i1} &= (r_{i0x} + \frac{l_{i1}}{2}\cos q_{i1})\,\vec{e}_x + (r_{i0y} + \frac{l_{i1}}{2}\sin q_{i1})\,\vec{e}_y \\
\vec{r}_{i2} &= (r_{i0x} + l_{i1}\cos q_{i1} + \frac{l_{i2}}{2}\cos q_{i2})\,\vec{e}_x + (r_{i0y} + l_{i1}\sin q_{i1} + \frac{l_{i2}}{2}\sin q_{i2})\,\vec{e}_y \\
\vec{r}_{\mathrm{ie}} &= (r_{i0x} + l_{i1}\cos q_{i1} + l_{i2}\cos q_{i2})\,\vec{e}_x + (r_{i0y} + l_{i1}\sin q_{i1} + l_{i2}\sin q_{i2})\,\vec{e}_y,
\end{aligned} \tag{1}$$

respectively, whereas (r_{i0x}, r_{i0y}) denote the coordinates of the i-th electric drive. By assumption the cross-sections of all rigid bodies are constant and the TCP mass m_e represents a particle. Note that the time dependencies of the coordinate functions are omitted for convenience without causing any confusion. Now the velocities can be calculated using the time derivatives of (1) with respect to the Newtonian reference frame. This yields

$$\vec{v}_{i1} = \frac{d}{dt}\vec{r}_{i1}, \quad \vec{v}_{i2} = \frac{d}{dt}\vec{r}_{i2} \quad \text{and} \quad \vec{v}_{ie} = \frac{d}{dt}\vec{r}_{\mathrm{ie}}. \tag{2}$$

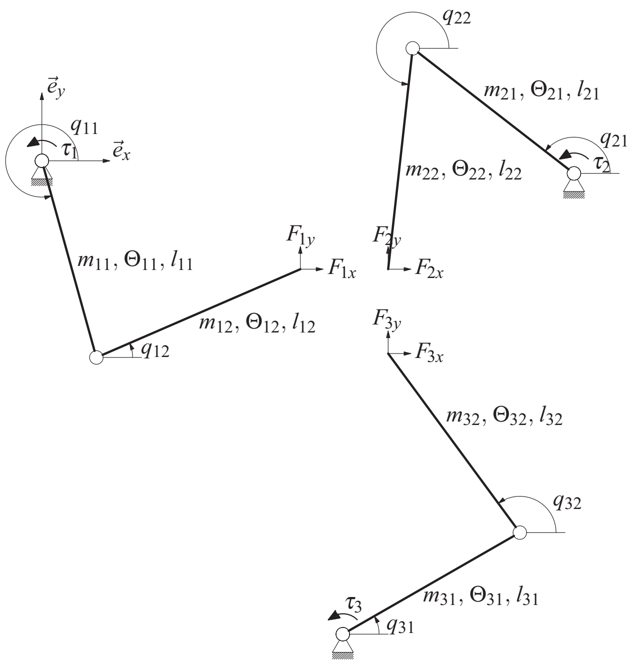

Figure 2: Free-body diagram of the planar redundant parallel robot

The derivation of the angular velocities is fairly simple and can be expressed in terms of the generalized coordinates q as[1]

$$\vec{\omega}_{i1} = \dot{q}_{i1}\,\vec{e}_z \quad \text{and} \quad \vec{\omega}_{i2} = \dot{q}_{i2}\,\vec{e}_z. \tag{3}$$

Defining the external forces $\vec{F}_i := F_{ix}\,\vec{e}_x + F_{iy}\,\vec{e}_y$ and the external torque vectors $\vec{\tau}_i := \tau_i\,\vec{e}_z$ the corresponding translational and angular velocities are directly given by $\vec{v}_{ie}$ and $\vec{\omega}_{i1}$ respectively. Lagrange's method can now be applied. The kinetic energy for any arm is given by

$$T_i = \sum_{j=1}^{2}\left(\frac{1}{2}\,m_{ij}\vec{v}_{ij}^{\,T}\vec{v}_{ij} + \frac{1}{2}\,\Theta_{ij}\vec{\omega}_{ij}^{\,T}\vec{\omega}_{ij}\right) + \frac{1}{2}\,m_{i\mathrm{e}}\vec{v}_{i\mathrm{e}}^{\,T}\vec{v}_{i\mathrm{e}} \tag{4}$$

with the TCP mass being evenly distributed among the three arms which yields $m_{i\mathrm{e}} := \frac{m_\mathrm{e}}{3}$. Since the potential energy vanishes, $U_i = 0$ and the Langrangian function becomes

$$L_i = T_i. \tag{5}$$

The equations of motion for any of the subsystems can now be computed by the famous Lagrange's equations

$$\frac{d}{dt}\left(\frac{\partial L_i}{\partial \dot{q}_{ij}}\right) - \frac{\partial L_i}{\partial q_{ij}} = Q_{ij}, \quad j = 1,2 \tag{6}$$

under consideration of the external forces

$$Q_{ij} = \frac{\partial \vec{\omega}_{i1}}{\partial \dot{q}_{ij}} \cdot \vec{\tau}_i + \frac{\partial \vec{v}_{ie}}{\partial \dot{q}_{ij}} \cdot \vec{F}_i. \tag{7}$$

Defining $q_i := \begin{bmatrix} q_{i1} & q_{i2} \end{bmatrix}^T$ and omitting the arrows indicating vectorial quantities the equations of motion are of the form

$$M_i(q_i)\ddot{q}_i + C_i(\dot{q}_i, q_i) = G_{i\mathrm{F}}(q_i)\,F_i + G_{i\tau}(q_i)\,\tau_i. \tag{8}$$

So far the derivation of the dynamics of the subsystems has been straight-forward. Exploiting Newton's third law $\sum_{i=1}^{3} F_i = 0$ which means that the joint forces are not independent. Therefore F_{3x} and F_{3y} can be expressed

[1] $\dot{(.)} = \frac{d}{dt}(.)$

in terms of F_{1x}, F_{1y}, F_{2x} and F_{2y} alone. Defining $\tau := \begin{bmatrix}\tau_1 & \tau_2 & \tau_3\end{bmatrix}^T, F := \begin{bmatrix}F_{1x} & F_{1y} & F_{2x} & F_{2y}\end{bmatrix}^T$ and $q := \begin{bmatrix}q_{11} & q_{12} & q_{21} & q_{22} & q_{31} & q_{32}\end{bmatrix}^T$, the equations of motions can be compressed to

$$M(q)\ddot{q} + C(\dot{q}, q) = G_{\mathrm{F}}(q)\,F + G_{\tau}(q)\,\tau. \tag{9}$$

with positive definit matrix $M(q)$. Note that this representation of the dynamics of the subsystems is affine with respect to the external forces and torques. In order to describe the dynamics of the overall system the holonomic constraints for the central joint have to be added. From $\vec{r}_{1\mathrm{e}} = \vec{r}_{2\mathrm{e}} = \vec{r}_{3\mathrm{e}}$ the four independent algebraic equations

$$\begin{aligned} 0 &= \vec{r}_{1\mathrm{e}}(q) - \vec{r}_{3\mathrm{e}}(q) \\ 0 &= \vec{r}_{2\mathrm{e}}(q) - \vec{r}_{3\mathrm{e}}(q) \end{aligned} \tag{10}$$

are chosen. Finally the mechanical model of the parallel robot is represented by (9) and (10).

2.2 State space representation

Using the definiteness of $M(q)$ and again omitting the arrows a differential-algebraic state space model

$$\begin{bmatrix}\dot{q}\\ \ddot{q}\end{bmatrix} = \begin{bmatrix}\dot{q}\\ -M(q)^{-1}\,C(\dot{q},q)\end{bmatrix} + \begin{bmatrix}0\\ M(q)^{-1}\,G_{\mathrm{F}}(q)\end{bmatrix} F + \begin{bmatrix}0\\ M(q)^{-1}\,G_{\tau}(q)\end{bmatrix}\tau \tag{11}$$

$$0 = \begin{bmatrix} r_{1\mathrm{e}}(q) - r_{3\mathrm{e}}(q)\\ r_{2\mathrm{e}}(q) - r_{3\mathrm{e}}(q)\end{bmatrix} \tag{12}$$

can easily be obtained. Denoting $x_1^T = \begin{bmatrix}q^T & \dot{q}^T\end{bmatrix}$, $x_2 = F$, $u = \tau$ and the observation that the algebraic equations (12) depend on q only yields the differential-algebraic normal form

$$\dot{x}_1 = a_{11}(x_1) + A_{12}(x_1)\,x_2 + B_1(x_1)\,u \tag{13}$$

$$0 = a_{21}(x_1). \tag{14}$$

2.3 Formulation of the control problem

Obviously the numerical simulation of the descriptor system (13), (14) is a nontrivial task. The key idea of this contribution is the combination of rewriting this set of differential-algebraic equations into a nonlinear coupling or constraint control problem respectively motivated by the approach in [5] for linear systems and designing an appropriate nonlinear feedback law which recovers the original dynamics of the robotic system. For being able to handle the algebraic part of the mathematical model, some new constraint or coupling output quantities

$$y_{\mathrm{c}} := a_{21}(x_1) \tag{15}$$

are defined. One of the tasks as far as controller design is concerned will be to bring those fictitious outputs to zero asymptotically. To accomplish this another new input vector

$$u_{\mathrm{e}} := \begin{bmatrix}x_2\\ u\end{bmatrix} \tag{16}$$

is defined. Since (13) is affine with respect to x_2 and u the new overall open loop system

$$\begin{aligned} \dot{x}_1 &= a_{11}(x_1) + A_{12}(x_1)x_2 + B_1(x_1)u \\ &= a_{11}(x_1) + \begin{bmatrix}A_{12}(x_1) & B_1(x_1)\end{bmatrix}\begin{bmatrix}x_2\\ u\end{bmatrix} \\ &=: a(x_1) + B(x_1)u_{\mathrm{e}} \end{aligned} \tag{17}$$

$$\begin{aligned} y_{\mathrm{c}} &= a_{21}(x_1) \\ &=: c(x_1) \end{aligned} \tag{18}$$

is again input affine. Hence the effort concerning the computation of the class of feedback law still remains quite moderate.

3 Feedback control design

For feedback control purposes the system description will be embedded into an differential geometric context. Equations (17), (18) are considered as a local representation of the system dynamics on a smooth manifold $\mathscr{M}$, which means that x_1 can be identified via diffeomorphic coordinate functions φ with a point $\varphi^{-1}(x_1) = p \in \mathscr{M}$ which belongs to $\mathscr{M}$. Furthermore suppose $\dim \mathscr{M} = r$, $x_2 \in \mathbb{R}^{p_c}$, $u \in \mathbb{R}^p$ and that the number of algebraic equations q_c equals p_c. The vectorfields $a(x_1)$ and $c(x_1)$ and also the matrix-valued mapping $B(x_1)$ will considered to be smooth and therefore continuously differentiable.

3.1 Feedback structure and control objective

Before appropriate control objectives can be defined, a suitable feedback structure will be chosen. Regarding the famous Lagrange's equations (6) or also Newton's second law of motion, it is obvious that external forces and torques always contribute to these kind of mechanical manipulators in an affine fashion. Consequently any minimal representation would be input affine since only such torques serve as actuating variables. We will therefore choose

$$u_e = \begin{bmatrix} x_2 \\ u \end{bmatrix} = r(x_1) + f(x_1)\,\tilde{u} \tag{19}$$

as feedback form because any control of this kind transforms an input affine open loop system again into an input affine closed loop system as can be checked easily.

Now, the design issue is to find an appropriate state feedback controller (19) which has to meet certain requirements as far as the closed loop dynamics are concerned. Equation (14) causes the trajectory of $x_1(t)$ to remain on an invariant submanifold $\mathscr{N} \subset \mathscr{M}$ of allowed configurations for all times t. This means that if this submanifold is reached once the solution $x_1(t)$ will always remain on $\mathscr{N}$, no matter what driving torques are chosen. Regarding the descriptor model (13), (14) inconsistent initial conditions will be pulled to $\mathscr{N}$ immediately. Clearly the control (19) law applied to the coupling control problem (17), (18) will not recover this impulsive behavior since the algebraic part does not appear in the closed loop. On the other hand such inconsistent initial conditions do not exist in reality and they are therefore rather part of the mathematical model than part of the physics behind. The physics of the system are thus given by equation (13) on $\mathscr{N}$ only.

Consequently the first control objective is to find a feedback law that renders the manifold of admissible states invariant which can be expressed as

$$\left.\begin{array}{l} a(x_1) + B(x_1)\,r(x_1) \subset T_{x_1}\mathscr{N} \\ c(x_1) = 0 \end{array}\right\} \forall x_1 \in \mathscr{N}, \tag{20}$$

whereas $T_{x_1}\mathscr{N}$ denotes the tangent space of $\mathscr{N}$. Clearly, for all admissible states and therefore for any x_1 belonging to $\mathscr{N}$ the fictitious outputs have to vanish to guarantee that the algebraic equations are not violated.

Even if the physical system lives on $\mathscr{N}$ only, for numerical purposes this submanifold is required to be globally attracting. Consequently any solution will be pulled to $\mathscr{N}$ whenever inconsistent initial conditions or numerical errors occur. Because of the physical reasons the way how this is achieved is of secondary importance.

The remaining degrees of freedom in the controller design procedure can now be used to make the system behave in a desired way on the invariant submanifold of allowed states. This can either be used in order to design a controller in the classical sense to solve some trajectory following problems or to simulate the robotic system without any exterior control. For the latter case the original dynamics on $\mathscr{N}$ have to be obtained which defines the last control objective.

In order not to excite any prohibited motion the choice of input variables has to be obviously such that

$$B(x_1)\,f(x_1) \subset T_{x_1}\mathscr{N} \quad \forall x_1 \in \mathscr{N}. \tag{21}$$

Summarizing the static feedback control (19) has to meet two requirements:

- The manifold $\mathscr{N}$ of admissible states has to be invariant and asymptotically attracting.
- For system simulation the dynamics on $\mathscr{N}$ have to be the original dynamics of the system in minimal representation.

3.2 Constraint dynamics algorithm

As far as the computation of a maximal controlled invariant and output-nulling submanifold and the corresponding feedback law is concerned the basic idea can be found in [7] for linear systems and is extended to nonlinear input affine systems in [2] and [8] resulting in the constraint dynamics algorithm. This approach can directly be addressed to the posed control problem such that the required invariance property of the manifold of consistent states is achieved for the closed loop dynamics. It is not quite clear which dimension this manifold is going to have since the number of degrees of freedom of a system including closed kinematic chains is not determined by the number and the nature of the joints alone. The assembly of the single joints is actually of great importance. Note, that the proposed algorithm, that will be used here provides precisely the maximal controlled invariant and output-nulling submanifold. Therefore any allowed motion of the parallel robotic manipulator will be kept in the derived simulation model.

Before sketching the constraint dynamics algorithm as described in [8], some mathematical framework is introduced. Let $F : \mathscr{M}_1 \to \mathscr{M}_2$ be a map between manifolds with dimensions $\dim(\mathscr{M}_i) = n_i$ and let $p_2 \in \mathscr{M}_2$. Suppose that $F^{-1}(p_2)$ is nonempty and $\mathrm{rank}(F)$ at every point of $F^{-1}(p_2)$ equals n_2. Then $F^{-1}(p_2)$ is a submanifold of $\mathscr{M}_1$ of dimension $n_1 - n_2$. The rank of F at point p is given by the rank of the jacobian calculated at $\varphi(p)$ whereas φ is a set of coordinate functions for M_1. Additionally $L_f h$ denotes the Lie-derivative and is given in local coordinates by $L_f h = \frac{\partial h(x)}{\partial x} f(x)$.

The constraint dynamics algorithm is now based on an successive nesting of submanifolds starting from $\mathscr{M}$. Consider the system (17), (18).

Step $k = 0$: By assumption the output vector field $c(x_1)$ contains q_c independent scalar functions $c_i(x_1)$, which means that the rank of $c(x_1)$ is maximal. Introducing restrictions

$$\phi_i(x_1) := c_i(x_1), \quad 1 \le i \le q_\mathrm{c} = n_0,$$

the first nested submanifold can be expressed as

$$\mathscr{N}^0 := \{x_1 \in \mathscr{M} \,|\, \phi_i(x_1) = 0,\, 1 \le i \le n_0\}.$$

$\vdots$

Step k: At the k-th step, the change of the restrictions ϕ_i along the controlled vectorfield $a(x_1) + B(x_1)\, r(x_1)$ is analyzed. Defining matrices $M_\mathrm{B}^k = (m_\mathrm{B}^k)_{ij}$ and $M_\mathrm{a}^k = (m_\mathrm{a}^k)_i$

$$\begin{aligned}(m_\mathrm{a}^k)_i(x_1) &:= L_{a(x_1)}\phi_i(x_1), \ \ 1 \le i \le \sum_{l=0}^{k-1} n_l \\ (m_\mathrm{B}^k)_{ij}(x_1) &:= L_{b_j(x_1)}\phi_i(x_1), \ \ 1 \le i \le \sum_{l=0}^{k-1} n_l, \ \ 1 \le j \le p_\mathrm{c} + p,\end{aligned}$$

whereas $b_j(x_1)$ denotes the j-th column of $B(x_1)$. Assuming that M_B^k has constant rank r_k in a neighborhood of $\mathscr{N}^{k-1}$, after a permutation of the constraints ϕ_i, it is certainly possible (by implicit function theorem) to find a feedback, such that

$$M_\mathrm{a}^k(x_1) + M_\mathrm{B}^k\, r(x_1) = \begin{bmatrix} 0_{r_k \times 1} \\ \psi_k(x_1) \end{bmatrix} \overset{!}{=} 0 \quad \text{on} \quad \mathscr{N}^{k-1}.$$

Let $\mathrm{rank}(\psi_k) = n_k$ again be constant on $\mathscr{N}^{k-1}$ by assumption. Therefore n_k independent elements of ψ_k can be chosen as further restrictions

$$\phi_i(x_1), \quad n_{k-1} < i \le n_{k-1} + n_k.$$

The new nested submanifold is definded as

$$\mathscr{N}^k := \left\{ x_1 \in \mathscr{M} \,\middle|\, \phi_i(x_1) = 0,\, 1 \le i \le \sum_{l=0}^{k} n_l \right\}.$$

$\vdots$

Step $k^\star$: Finally the algorithm determines when

$$M_\mathrm{a}^{k^\star}(x_1) + M_\mathrm{B}^{k^\star}(x_1)\, r(x_1) = 0 \quad \text{on} \quad \mathscr{N}^\star =: \mathscr{N} \quad \Leftrightarrow \quad \psi_{k^\star} = 0. \tag{22}$$

A point x_{10} is called regular point of the algorithm, if both rank assumptions are satisfied.

The requirement with respect to the feedforward gain (21) can easily be translated into a constructive version and yields

$$M_{\mathrm{B}}^{k^\star}(x_1)\,f(x_1) = 0, \tag{23}$$

whereas the number of independent control inputs just equals the dimension of the kernel of $M_{\mathrm{B}}^{k^\star}(x_1)$.

3.3 Making the manifold of admissible states attractive

Introducing some modifications it is furthermore possible to make the invariant manifold $\mathscr{N}$ an attracting submanifold, such that all inconsistent initial conditions tend asymptotically towards $\mathscr{N}$. This is especially important for simulation purposes. Technically the modification of the constraint dynamics algorithm is done by slightly changing the feedback law $r(x_1)$ or using the remaining degrees of freedom respectively without losing the invariance property of $\mathscr{N}$. Regarding equation (22) which plays a key role in the controller design the right hand side vanishes on $\mathscr{N}$ only. Thus any combination of the restrictions ϕ_i may be added to the first $r_{k^\star}(x_1)$ equations since the invariant submanifold is just defined by those restrictions which have to vanish on the latter. This yields

$$M_{\mathrm{a}}^{k^\star}(x_1) + M_{\mathrm{B}}^{k^\star}(x_1)\,r(x_1) = \begin{bmatrix} \sum_{i=1}^{n} p_i(x_1)\,\phi_i(x_1) \\ 0 \end{bmatrix} \quad \text{on} \quad \mathscr{M} \quad \text{with} \quad n := \sum_{l=0}^{k^\star-1} n_l. \tag{24}$$

Because of the properties of $M_{\mathrm{B}}^{k^\star}(x_1)$ solvability is still guaranteed. Note, that the coefficients p_i, that are to be chosen, stand for $r_{k^\star}$-dimensional vectors. Furthermore the calculation of the restrictions $\phi_i(x_1)$ during the algorithm will not change which certainly results in the same $\mathscr{N}$.

Making the obtained maximal invariant manifold attracting by the proposed structure might be a nontrivial task or even impossible in general. Nevertheless concerning plenty of relevant applications, the actual calculation reveals major simplifications. If the first appearance of any of the fictitious output quantities $c_i(x_1)$ or their time derivatives in $M_{\mathrm{B}}^{k}(x_1)$ leads to a new independent row, no further restrictions $\phi_i(x_1)$ will be generated. In this case the restrictions directly denote the output variables $c_i(x_1)$ and their time derivatives and thus a pole-placement procedure can be carried out, leading to exponential stability of the fictitious output variables which hence tend to zero.

Choosing an appropriate set of local coordinates, for the closed loop system

$$\begin{aligned} \dot{x}_1 &= a(x_1) + B(x_1)\,r(x_1) + B(x_1)\,f(x_1)\tilde{u} \\ y_{\mathrm{c}} &= c(x_1) \end{aligned} \tag{25}$$

the coupling feedback law yields the triangular structure

$$\begin{aligned} \dot{\tilde{x}}_1 &= \tilde{a}_1(\tilde{x}_1, \tilde{x}_2) + \tilde{B}_1(\tilde{x}_1, \tilde{x}_2)\tilde{u} \\ \dot{\tilde{x}}_2 &= \tilde{a}_2(\tilde{x}_2) \\ y_{\mathrm{c}} &= \tilde{c}(\tilde{x}_2). \end{aligned}$$

Obviously all the states $\tilde{x}_1$ have been rendered unobservable with respect to the fictitious coupling output variables y_{c}. In contrast $\tilde{x}_2$ is uncontrollable and therefore the subsystem $\dot{\tilde{x}}_2 = \tilde{a}_2(\tilde{x}_2)$ has to be stabilized by the proposed feedback.

Considering the presented example of the parallel robotic manipulator the second time derivative, which means the acceleration, can be influenced directly by the extended input vector u_{e} for any of the holonomic constraint equations. Exemplarily if $\phi_{i0} = c_i$ then $\frac{d}{dt}c_i$ is denoted by ϕ_{i1}. Since $\frac{d}{dt}\phi_{i1}$ can be controlled directly in an independent manner, applying the proposed feedback law $\frac{d^2}{dt^2}c_i$ can be expressed in terms of c_i and $\frac{d}{dt}c_i$. Thus for the decaying of the error defined by c_i two poles may be assigned and hence exponential stability can be guaranteed. From the physical point of view this is quite intuitive, since certainly there exist joint forces keeping the specific joints together.

The proposed procedure is similar to the well-known case of output-linearization, but it should be emphasized that in the case of the design of coupling feedback laws the system in general does neither have to be output-linearizable nor has to have stable zero dynamics. The requirements are therefore significantly less demanding.

3.4 Recovery of the open loop dynamics on $\mathscr{N}$

In [8] the inequality $r_{k^\star} \leq p_{\mathrm{c}}$ is proven, that is there are at most p_{c} independent rows contained in $M_{\mathrm{B}}^{k^\star}$. Since the fictitious input vector u_{e} is of dimension $p + p_{\mathrm{c}}$ the feedback $r(x_1)$ and therefore some degrees of freedom remain to be assigned. The goal will be to leave the open loop dynamics of the nonlinear descriptor system unchanged on $\mathscr{N}$.

For this purpose the feedback law is devided into two summands

$$r(x_1) = \underbrace{\begin{bmatrix} r_{11}(x_1) \\ 0 \end{bmatrix}}_{r_1(x_1)} + \underbrace{\begin{bmatrix} r_{12}(x_1) \\ r_{22}(x_1) \end{bmatrix}}_{r_2(x_1)} \tag{26}$$

with corresponding matrix

$$M_{\mathrm{B}}^{k^\star}(x_1) = \begin{bmatrix} M_{\mathrm{B1}}^{k^\star}(x_1) & M_{\mathrm{B2}}^{k^\star}(x_1) \end{bmatrix}$$

from (24), whereas $M_{\mathrm{B1}}^{k^\star}(x_1)$ and $M_{\mathrm{B2}}^{k^\star}(x_1)$ determine the contribution of the fictitious inputs x_2 and the primary input variables u respectively. Suppose that $M_{\mathrm{B}}^{k^\star}(x_1)r_2(x_1) = 0$, $\forall x_1 \in \mathscr{N}$. Thus the second part of (26) does not influence the dynamics on $\mathscr{N}$ and may be used for instance to make this manifold attracting. Hence the dynamics on $\mathscr{N}$ are determined by $r_1(x_1)$ only. Since the input variables u belonging to the original descriptor system are assumed to be chosen independently on $\mathscr{N}$ at any time the feedback for system simulation must not use them as input variables. This yields the specific structure of $r_1(x_1)$ with $r_{21}(x_1) = 0$ which has to be chosen for system simulation. For the robotic manipulator $r_{k^\star} = p_{\mathrm{c}}$ is actually fulfilled. Furthermore $M_{\mathrm{B1}}^{k^\star}(x_1)$ is not only quadratic but also regular. So the coupling control $r(x_1)$ can be written as feedback using x_2 as input variables only. As a direct consequence the vector-valued function $r_{11} : \mathscr{M} \to \mathbb{R}^{p_{\mathrm{c}} \times 1}$ and hence the dynamics on the invariant manifold of admissible states are uniquely determined. Since there are no further degrees of freedom for $r(x_1)$ on $\mathscr{N}$, the resulting dynamics have to be just the open loop dynamics of the original differential-algebraic system. Additionally because $M_{\mathrm{B1}}^{k^\star}(x_1)$ is regular $\mathrm{Im}\left(M_{\mathrm{B2}}^{k^\star}(x_1)\right) \in \mathrm{Im}\left(M_{\mathrm{B1}}^{k^\star}(x_1)\right)$ is implied. Consequently a feedforward gain of the form

$$f(x_1) = \begin{bmatrix} f_1(x_1) \\ f_2(x_1) \end{bmatrix} = \begin{bmatrix} f_1(x_1) \\ \mathrm{I} \end{bmatrix}$$

may always be chosen. Therefore the influence $\tilde{u}$ on the controlled system precisely equals the influence of u on system (13),(14) as long as $x_1 \in \mathscr{N}$. This can also be written as $\tilde{u} = u$ for the set of admissible states.

So finally under some rank assumptions the required feedback gain for system simulation could be found by an appropriate structural constraint given in (26).

3.5 Feedback control on $\mathscr{N}$

The derived closed loop model for system simulation (25) can now be used for further system analysis and classical feedback design. Note, that this is indeed significantly easier than dealing with the original set of differential-algebraic equations. For the examined parallel robotic manipulator $\dim \mathscr{N} = 4$ can be obtained. Physically this is quite intuitive. Considering a fixed pose, which means that the elbow links will not be stretched during operation, the kinematics are completely determined by the (x,y)-coordinates of the TCP. Accordingly the mechanical system has two degrees of freedom which yields a state space representation of dimension four on $\mathscr{N}$.

The following idea is to achieve a classical feedback linearization (see [2]) on the invariant manifold of admissible states. Therefore two purely kinematic control variables

$$\begin{aligned} y_1 &= \vec{r}_{1\mathrm{e}} \cdot \vec{e}_x = l_{11} \cos q_{11} + l_{12} \cos q_{12} \\ y_2 &= \vec{r}_{1\mathrm{e}} \cdot \vec{e}_y = l_{11} \sin q_{11} + l_{12} \sin q_{12} \end{aligned} \tag{27}$$

are defined intended for positioning the TCP. Obviously since there are three independent input variables τ_1, τ_2, τ_3 the problem is non quadratic. The examination of the control variables (27) yields that they are independent and have relative degree two each, such that by an appropriate choice of local coordinates the dynamics on $\mathscr{N}$ can be defined by these two outputs. Therefore another third control variable of relative degree zero has to be added. Note, that the zero dynamics of the twelfth order simulation model are rendered stable by the inner feedback loop for system simulation.

As third control variable hence

$$y_3 = \tau_1 + \tau_2 + \tau_3 \tag{28}$$

is defined which allows prestressing of the manipulator. This is quite important and simultaneously one of the most significant advantages of such a type of robot since position accuracy may be increased. Note, that (28) contains another input affine output equation.

Finally an exact input-output linearization can be carried out allowing the placement of four poles on the configuration space of the planar redundant robotic manipulator. As far as the original descriptor system is concerned no such zero dynamics are going to appear.

4 Simulation results

Considering the robotic example some numerical simulation results will be presented. The maximal invariant manifold $\mathscr{N}$ has dimension four and is embedded into an 12-dimensional state space. Since the mentioned assumptions

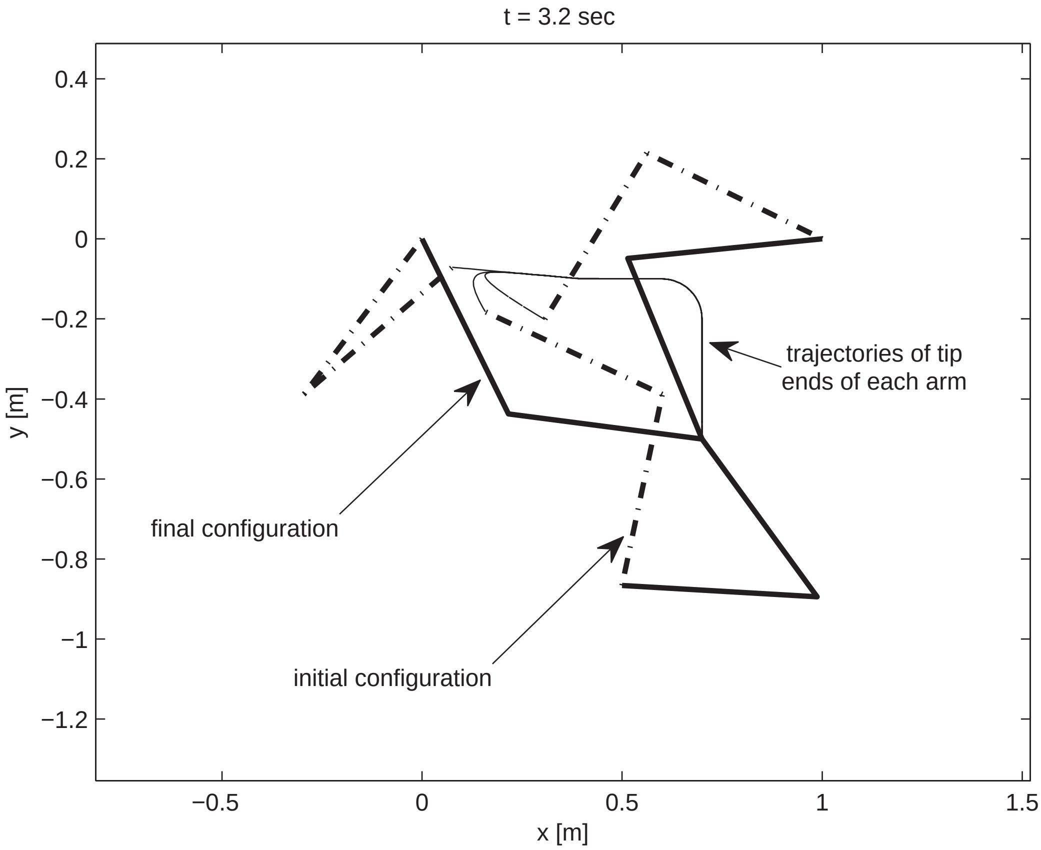

Figure 3: Numerical integration of the controlled system

for applying the constraint dynamics algorithm are satisfied this manifold is made attractive by the proposed feedback law for system simulation. For that purpose all eight poles are put to $\lambda_{ic} = -25$ for $i = 1, \ldots, 8$. In addition a classical state feedback controller is designed on $\mathscr{N}$ based on the exact input-output linearization procedure. The corresponding poles are located at $\lambda_{i\mathrm{fb}} = -5$ for $i = 1, \ldots, 4$. As depicted in figure 3 from initially inconsistent conditions all tip ends of the three arms are pulled to one point, which means that the trajectory tends to the set of admissible states. Exemplarily this is again shown in figure 4 for the coupling errors with respect to two of the four holonomic equations, namely $(\vec{r}_{3\mathrm{e}} - \vec{r}_{1\mathrm{e}}) \cdot \vec{e}_x$ and $(\vec{r}_{3\mathrm{e}} - \vec{r}_{1\mathrm{e}}) \cdot \vec{e}_y$. Note, that once $\mathscr{N}$ is reached independently of any chosen input variable vector it will never be left. In the following a certain trajectory consisting of a straight line, a curve of 90 degrees and another straight line is tracked by an appropriate feedforward control. The results are not surprising since no disturbances are appearing. In addition after two seconds the structure is prestressed by applying an offset of -10Nm to the sum of the three driving torques. This results in the joint forces acting on the first arm which change significantly after two seconds as can be seen in figure 5.

5 Summary and conclusion

In this contribution a systematic approach of modelling, simulation and control of a class of nonlinear descriptor systems has been derived. As far as mechanical systems are concerned rather than eliminating auxiliary coordinates or substituting holonomic by nonholonomic bindings the differential-algebraic equations including the holonomic bindings are used directly as starting point for further system analysis. Hence the overall mechanical system can be divided into smaller parts which simplifies the modelling procedure significantly.

Introducing some additional, fictitious input and output variables the differential-algebraic model can be interpreted as a coupling or constraint control problem. Under some regularity assumptions the constraint dynamics algorithm may be exploited to compute the maximal controlled invariant submanifold of admissible states for the mechanical model including a constructive way of designing an appropriate feedback law. Even though further investigations are necessary it seems that the mentioned regularity assumptions are always fulfilled as long as the mechanical problem formulation of the descriptor system is well-posed. Considering the elements of the cut open system having tree structure this means that the number of the defined independent joint forces or torques are suitable of keeping the respective joint together.

However if the regularity assumptions are satisfied the proposed procedure does not only allow a numerical reli-

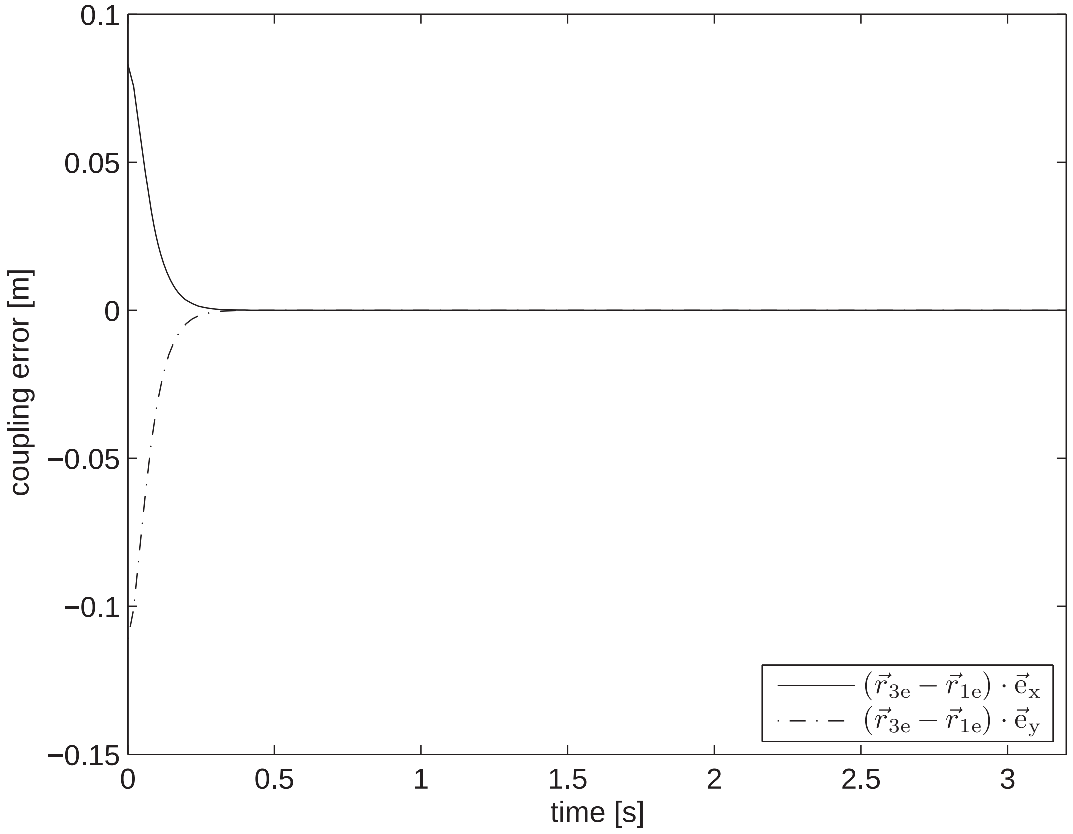

Figure 4: Coupling error between 1st and 3rd arm

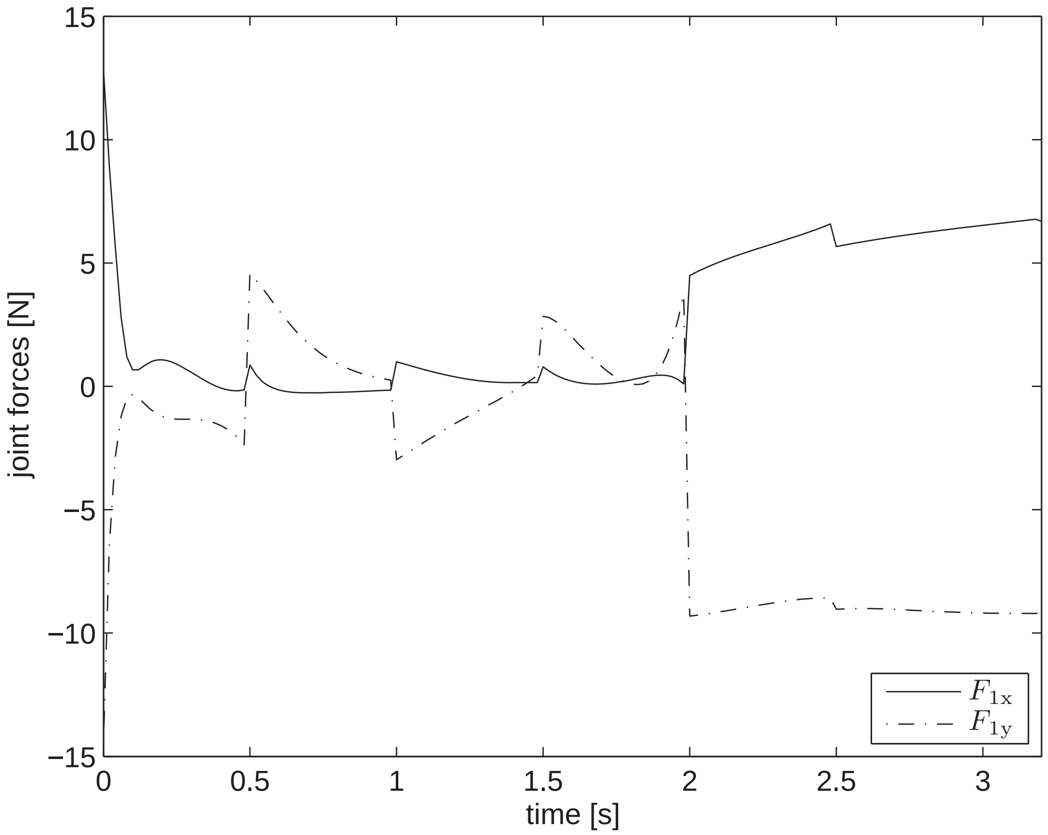

Figure 5: Joint forces acting on 1st arm

able method for system simulation with standard integration tools, but also permits a natural physical inside into the mechanical system since the number of degrees of freedom of the can directly be checked considering the dimension of the maximal controlled invariant manifold which renders the defined fictitious output variables zero. Furthermore by some modifications to the feedback law this manifold can be made an attractive set such that standard integration methods may be used even if inconsistent initial conditions appear.

The remaining degrees of freedom in the controller design procedure can be used to make the system behave in a desired way on the submanifold of allowed states. This can either be used in order to design a controller in the classical sense to solve some trajectory following problems or to simulate the robotic system without any exterior control. The latter control object is achieved by introducing some structural constraints into the feedback law. This feedback just recovers the finite dynamics of the original differential-algebraic system on submanifold of admissible states and is therefore a suitable tool for the numerical stable simulation.

The procedure has been exemplarily carried out in order to analyze the rather involved and nonlinear dynamics of a planar redundant parallel robot.

For future works and for deeper understanding the solvability of the problem should be referred to basic properties of the differential-algebraic model such as formal integrability for instance. Furthermore it is desirable that direct feedthrough can be included in an input affine or eventually even in arbitrary fashion in order to expand the class of treatable systems. This is especially important to include constitutive equations such as material laws, springs or dampers into this way of modelling.

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