STABILIZATION OF NONLINEAR SYSTEMS WITH INPUT SATURATION USING SOS-PROGRAMMING

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Abstract. This paper presents a numerical approach to design a nonlinear control law for nonlinear systems by extending the results for controller design using SOS-programming (see e.g. [4]) to systems with input saturation. In addition, a Lyapunov function is constructed which is used to prove the stability and to estimate the region of attraction of the equilibrium point of the nonlinear closed loop system. Considering the input saturation yields additional conditions for the SOS-program. A second modification is the extension to rational control laws which can be integrated in the SOS-programming method by using a polynomial nominator and denominator. In the end, two different SOS-programs for the construction of a control law and a Lyapunov function of a polynomial system with input saturation are derived and an estimation of the region of attraction of the stabilized equilibrium point is computed. An academic example demonstrates the presented approach.

1 Introduction

The stabilization of an equilibrium point of a nonlinear system is an important problem in control theory. The input saturation of a system has significant influence on the stabilization task, thus it is an advantage to consider the input saturation systematically within the design procedure. Furthermore, a proof of stability of the equilibrium of the nonlinear system with input saturation and an estimation of the region of attraction are the interesting properties in the design approach.
Although, the Lyapunov theory is the instrument to give the answer to the stability question (see e.g. [5]), there is no easy way of constructing an appropriate Lyapunov function. In the last decade, beginning with the work of Parrilo [6], a numerical approach to find a Lyapunov function was presented in different publications, e.g. [4] and [9], using the so called SOS-programming (SOS - sum of square). It is shown in [6] that every SOS-program can be transfered in a semidefinite program which can be solved efficiently by a numerical software package. Furthermore, the region of attraction of the considered equilibrium point can be estimated using SOS-programming. The basic problem finding a suitable Lyapunov function is to test nonlinear functions for positive definiteness on a certain region in state space. The advantage of the SOS-programming technique is to solve this problem for polynomial functions with a restriction. Actually the polynomial is checked if a sum of square representation exists which is naturally positive semidefinite, for details see e.g. [6] and [8]. Furthermore, the SOS-programming technique can construct Lyapunov functions of degree larger than 2 systematically which may result in better estimates of the region of attraction when compared to quadratic Lyapunov functions.
The Lyapunov theory can also be used for the design of stabilizing control laws which yields the computation of an appropriate control law and a suitable Lyapunov function which can be used for an estimation of the region of attraction of the equilibrium point of the closed loop system. The SOS-programming approach for the stabilization of nonlinear systems in [4] is extended to systems with input saturation which yields two additional conditions in the SOS-programs for the determination of the control law and the Lyapunov function. Therefore, the degrees of freedom contained in the nonlinear control law are used in the SOS-programs to construct a suitable Lyapunov function for providing the stability and to determine an as large as possible estimation of the region of attraction of the equilibrium point of the closed loop system. The idea of considering the input saturation for the purpose of an estimation of the region of attraction for a given control law and Lyapunov function using SOS-programming was used in [1]. In difference, the presented approach embeds the constraint of the input saturation in the whole procedure and consequently leads to the construction of a control law and a Lyapunov function where both try to adopt the shape of the exact region of attraction of the equilibrium point.
Another modification compared to [4] is the application of rational control laws which consist of polynomial nominators and denominators. Furthermore, the denominator is chosen such that it is strictly positive in the whole state space in order to avoid singularities in the input signal. The integration of a rational control law in the constraints of the optimization problem yield after small rearrangements polynomial constraints which can be solved with the SOS-programming technique.

In the next section an introduction of the problem is given. A brief overview of the SOS-programming is stated in Section 3. In Section 4 two different algorithms for the design procedure are derived, consisting of the systematic construction of a stabilizing control law and a suitable Lyapunov function together with an estimation of the region of attraction of the equilibrium point. An example in Section 5 demonstrates the proposed approach.

2 Problem formulation

For the stabilization problem consider the nonlinear system

$$\dot{x} = f(x) + g(x)u \tag{1}$$

with the state $x \in \Omega \subseteq \mathbb{R}^n$, where Ω is a domain such that $0 \in \Omega$ and the input $u \in \mathbb{R}$. In the sequel, the elements of f and g are supposed to be from the set $\mathbb{R}[x]$ of multivariable polynomials in the n variables x with coefficients in $\mathbb{R}$. Furthermore, let without loss of generality $x = 0$ for $u = 0$ be an equilibrium point of (1), i.e. $f(0) = 0$ and $g(0) \neq 0$, and in addition the input of system (1) is restricted by an lower and upper constant input saturation, i.e.

$$u_{lo} \leq u \leq u_{up} \tag{2}$$

The aim of the control design is the stabilization of the equilibrium point $x = 0$ with a control signal u that satisfies (2). To this end consider the rational control law

$$u = -\frac{q(x)}{r_0 + r(x)}, \quad r_0 > 0 \tag{3}$$

with $q(0) = 0$ and polynomial functions in the nominator and denominator of the the control law (3), i.e. $q(x),\ r(x) \in \mathbb{R}[x]$, furthermore, the denominator is supposed to be positive semidefinite, i.e. $r(x) \geq 0,\ \forall x \in \mathbb{R}^n$, in order to avoid singularities in the control law (3). Introducing (3) into (1) yields the closed loop system

$$\dot{x} = f(x) - g(x)\frac{q(x)}{r_0 + r(x)} \tag{4}$$

with an equilibrium point at $x = 0$. The further stabilization analysis which is used in the control design, is based on Lyapunov's stability theorem for asymptotic stability of the equilibrium point of (4) without consideration of the input saturation (2) (see e.g. [5]).

Theorem 1 *Let $x = 0$ be an equilibrium point for* (4) *and $D \subseteq \Omega$ be a domain containing $x = 0$. Let $V(x) : D \to \mathbb{R}$ be a continuously differentiable function such that*

$$V(x) > 0, \qquad \forall x \in D \setminus \{0\} \text{ and } V(0) = 0 \tag{5}$$

$$\dot{V}(x) = \frac{\partial V(x)}{\partial x}\left(f(x) - g(x)\frac{q(x)}{r_0 + r(x)}\right) < 0, \quad \forall x \in D \setminus \{0\} \text{ and } \dot{V}(0) = 0 \tag{6}$$

Then, the equilibrium point $x = 0$ is asymptotically stable and $V(x)$ is a Lyapunov function *of the closed loop system* (4).

As it can be seen in (5)–(6) the basic problem using Theorem 1 is to find a stabilizing control law (3), a suitable Lyapunov function $V(x)$ and check nonlinear functions for positive definiteness. This task can be accomplished by using the SOS-programming technique since only rational terms enter (5)–(6) (see [4] and [6]). Furthermore, the condition (2) must be satisfied within the whole procedure of constructing a control law and a Lyapunov function, which yields two additional conditions

$$\frac{q(x)}{r_0 + r(x)} \leq u_{up} \tag{7}$$

$$\frac{q(x)}{r_0 + r(x)} \geq u_{lo} \tag{8}$$

concerning the control law (3). Taking these two conditions (7)–(8) into account by the construction of the control law (3) and a Lyapunov function by using Theorem 1, asymptotic stability of the equilibrium point $x = 0$ of the closed loop system (4) with input saturation (2) is ensured. The conditions (5)–(6) and (7)–(8) can be formulated within the SOS-programming method which can be applied to solve the problem of constructing a control law that satisfies (2) and a Lyapunov function. Furthermore, the Lyapunov function can be used for the estimation of the region of attraction of the equilibrium point $x = 0$ using SOS-programming (see [4]). The advantage of the presented control design is the stabilization of the equilibrium point by a systematic construction of a Lyapunov function of degree larger than 2 using a numerical procedure. This may result in better estimation of the region of attraction of the equilibrium point when compared to quadratic Lyapunov functions. In addition the rational control law (3) represents a larger class of control laws compared to the polynomials used in [4].

3 Fundamentals of SOS-programming

3.1 SOS-polynomials

Before the implementation of the SOS-program is stated a brief introduction to the SOS-programming method itself will help to clarify the further proceeding.

An important subset of the polynomials $\mathbb{R}[x]$ are the sum of square (SOS) polynomials. The set of all *SOS-polynomials* in the n variables x is defined as

$$\Sigma[x] := \left\{ s \in \mathbb{R}[x] \middle| s = \sum_{i=1}^{k} p_i^2,\ p_i \in \mathbb{R}[x],\ k < \infty \right\} \tag{9}$$

Based on the definition of SOS-polynomials in (9), it is obvious that the relation $p(x) \geq 0,\ \forall x \in \mathbb{R}^n$ for a given polynomial $p(x) \in \mathbb{R}[x]$ holds, if a sum of square representation exists, i.e. $p(x) \in \Sigma[x]$. The advantage of this representation is, that there exists an efficient method to test a polynomial if it is a SOS-polynomial and, in the end, if it is positive semidefinite. Parrilo and co-workers created a MATLAB toolbox called SOSTOOLS (see [7]) whose basic task is to check polynomials for a SOS representation.

3.2 Positivstellensatz

In order to formulate the Positivstellensatz a few definitions have to be introduced.

Definition 1 *Given $\{g_1, \ldots, g_\sigma\} \in \mathbb{R}[x]$, the* multiplicative monoid *generated by the polynomials g_i is the set of all finite products of the polynomials g_i, including the empty product, defined to be* 1. *It is denoted as $\mathcal{M}(g_1, \ldots, g_\sigma)$. For completeness define $\mathcal{M}(\emptyset) := 1$.*

Example: $\mathcal{M}(g_1, g_2) = \left\{ g_1^{k_1} g_2^{k_2} \,|k_1, k_2 \in \mathbb{Z}_+ \right\}$

Definition 2 *Given $\{f_1, \ldots, f_\rho\} \in \mathbb{R}[x]$, the* cone *generated by the polynomials f_i is*

$$\mathcal{P}\left(f_1, \ldots, f_\rho\right) := \left\{ s_0 + \sum_{i=1}^{k} s_i b_i \middle| s_i \in \Sigma[x],\ b_i \in \mathcal{M}\left(f_1, \ldots, f_\rho\right) \right\} \tag{10}$$

For completeness note that $\mathcal{P}_n(\emptyset) := \Sigma[x]$.

Example:

$$\mathcal{P}(f_1, f_2) = \{s_1 + s_2 f_1 + s_3 f_2 + s_4 f_1 f_2 \,|s_1, \ldots, s_4 \in \Sigma[x]\} \tag{11}$$

It is essential, that the product sf^2 of a polynomial $f \in \mathbb{R}[x]$ and a SOS-polynomial $s \in \Sigma[x]$ is a SOS-polynomial, i.e. $sf^2 \in \Sigma[x]$, thus every cone can be written as a sum of 2^ρ terms. This means for the example (11), e.g. the term $sf_1^2 f_2$ is contained in $s_3 f_2$.

Definition 3 *Given $\{h_1, \ldots, h_\tau\} \in \mathbb{R}[x]$, the* ideal *generated by the polynomials h_i is*

$$\mathcal{I}(h_1, \ldots, h_\tau) := \left\{ \sum_{k=1}^{\tau} h_k p_k \middle| p_k \in \mathbb{R}[x] \right\} \tag{12}$$

For completeness note that $\mathcal{I}(\emptyset) := 0$.

With these definitions the following theorem taken from [2] can be stated.

Theorem 2 (Positivstellensatz) *Given polynomials $\{f_1, \ldots, f_\rho\}$, $\{g_1, \ldots, g_\sigma\}$ and $\{h_1, \ldots, h_\tau\}$ in $\mathbb{R}[x]$ the following statements are equivalent:*

1. *The set*

$$\left\{ x \in \mathbb{R}^n \middle| \begin{array}{l} f_1(x) \geq 0, \ldots, f_\rho(x) \geq 0, \\ g_1(x) \neq 0, \ldots, g_\sigma(x) \neq 0, \\ h_1(x) = 0, \ldots, h_\tau(x) = 0 \end{array} \right\} \tag{13}$$

   is empty.

2. *There exist polynomials $f \in \mathcal{P}(f_1, \ldots, f_\rho)$, $g \in \mathcal{M}(g_1, \ldots, g_\sigma)$ and $h \in \mathcal{I}(h_1, \ldots, h_\tau)$ such that*

$$f(x) + g^2(x) + h(x) = 0 \tag{14}$$

By using Theorem 2 in the following section, the SOS-program for the stabilization and the estimation of the region of attraction can be derived.

4 Stabilization of nonlinear systems with input saturation

In order to obtain a systematic construction of a control law and a Lyapunov function, Theorem 1 and the restriction of the input saturation (7)–(8) are implemented in a SOS-program. The presented approach is an extension of the control design in [4] where two different algorithms are shown, thus two different SOS-programs are derived in the next subsections.

4.1 Optimization of the upper bound

Construction of a Control Law and a Lyapunov Function The first algorithm is following the method of the so-called "expanding D algorithm" in [4]. First of all a polynomial description of the domain D is defined by

$$D := \{x \in \mathbb{R}^n \,|\, p(x) \leq \beta\} \tag{15}$$

with a positive definite polynomial $p(x) > 0$, $\forall x \in \mathbb{R}^n \setminus \{0\}$, and $\beta \geq 0$ to ensure that D is connected and contains the equilibrium point $x = 0$. To provide the stability of the origin the conditions (5)–(6) of Theorem 1 and the conditions (7)–(8) must be met within the domain D, which read

$$D \setminus \{0\} \subseteq \{x \in \mathbb{R}^n \,|\, V(x) > 0\} \tag{16}$$
$$D \setminus \{0\} \subseteq \left\{x \in \mathbb{R}^n \,\middle|\, \frac{\partial V(x)}{\partial x}\left(f(x) - g(x)\frac{q(x)}{r_0 + r(x)}\right) < 0\right\} \tag{17}$$
$$D \subseteq \left\{x \in \mathbb{R}^n \,\middle|\, \frac{q(x)}{r_0 + r(x)} \leq u_{up}\right\} \tag{18}$$
$$D \subseteq \left\{x \in \mathbb{R}^n \,\middle|\, \frac{q(x)}{r_0 + r(x)} \geq u_{lo}\right\} \tag{19}$$

As already mentioned in Section 3, only polynomial inequalities can be handled in a SOS-program, thus conditions (17)–(19) have to be reformulated. Therefore a little restriction for the denominator polynomial $r(x)$ has to be made. In order to avoid singularities in the control law (3) the denominator polynomial $r(x)$ shall be a SOS-polynomial (i.e. $r(x) \in \Sigma[x]$), thus it is positive semi definite (i.e. $r(x) \geq 0$). Consequently, the denominator $r_0 + r(x)$ is positive definite in the whole state space because of the positive constant r_0 in the denominator. In the end, the inequalities in the three conditions (17)–(19) can be multiplied by the denominator polynomial which yields

$$D \setminus \{0\} \subseteq \{x \in \mathbb{R}^n \,|\, V(x) > 0\} \tag{20}$$
$$D \setminus \{0\} \subseteq \left\{x \in \mathbb{R}^n \,\middle|\, \frac{\partial V(x)}{\partial x}\left(f(x)(r_0 + r(x)) - g(x)q(x)\right) < 0\right\} \tag{21}$$
$$D \subseteq \{x \in \mathbb{R}^n \,|\, q(x) \leq u_{up}(r_0 + r(x))\} \tag{22}$$
$$D \subseteq \{x \in \mathbb{R}^n \,|\, q(x) \geq u_{lo}(r_0 + r(x))\} \tag{23}$$

The main intention of the approach is a good estimation of the region of attraction of the equilibrium point, so the domain D fulfilling the conditions (20)–(23) must be as large as possible. In other words an optimization problem looking for the maximum of β subject to the constraints (20)–(23) has to be solved. For using the SOS-programming method (20)–(23) have to be reformulated to get a polynomial for each condition which has to be checked for a SOS-representation. This can be achieved by using the Positivstellensatz (see Theorem 2), therefore the optimization problem with the constraints (20)–(23) written as empty sets is

$$\max \beta \quad \text{s.t.} \tag{24}$$
$$\{x \in \mathbb{R}^n \,|\, \beta - p(x) \geq 0,\ x \neq 0,\ -V(x) \geq 0\} = \emptyset \tag{25}$$
$$\left\{x \in \mathbb{R}^n \,\middle|\, \beta - p(x) \geq 0,\ x \neq 0,\ \frac{\partial V(x)}{\partial x}\left(f(x)(r_0 + r(x)) - g(x)q(x)\right) \geq 0\right\} = \emptyset \tag{26}$$
$$\left\{x \in \mathbb{R}^n \,\middle|\, \beta - p(x) \geq 0,\ q(x) - u_{up}(r_0 + r(x)) \geq 0,\ q(x) - u_{up}(r_0 + r(x)) \neq 0\right\} = \emptyset \tag{27}$$
$$\left\{x \in \mathbb{R}^n \,\middle|\, \beta - p(x) \geq 0,\ -q(x) + u_{lo}(r_0 + r(x)) \geq 0,\ -q(x) + u_{lo}(r_0 + r(x)) \neq 0\right\} = \emptyset \tag{28}$$

For applying the Positivstellensatz, all inequalities have to be polynomial (see (13)), but $x \neq 0$ in (25)–(26) is not, so these have to be replaced by positive definite polynomials $l_i(x) > 0$, $\forall x \in \mathbb{R}^n \setminus \{0\}$ and $i = 1, 2$, since

$$x \neq 0 \iff l_i(x) \neq 0 \tag{29}$$

This leads to the optimization problem

$$\max \beta \quad \text{s.t.} \tag{30}$$
$$s_1 + (\beta - p)s_2 - V s_3 - V(\beta - p)s_4 + l_1^{2k_1} = 0 \tag{31}$$
$$s_5 + (\beta - p)s_6 + \frac{\partial V}{\partial x}\left(f(r_0 + r) - gq\right)s_7 + \frac{\partial V}{\partial x}\left(f(r_0 + r) - gq\right)(\beta - p)s_8 + l_2^{2k_2} = 0 \tag{32}$$
$$s_9 + (\beta - p)s_{10} + (q - u_{up}(r_0 + r))s_{11} + (q - u_{up}(r_0 + r))(\beta - p)s_{12} + (q - u_{up}(r_0 + r))^{2k_3} = 0 \tag{33}$$
$$s_{13} + (\beta - p)s_{14} + (-q + u_{lo}(r_0 + r))s_{15} + (-q + u_{lo}(r_0 + r))(\beta - p)s_{16} + (-q + u_{lo}(r_0 + r))^{2k_4} = 0 \tag{34}$$

with $s_1,\dots,s_{16} \in \Sigma[x]$ and $k_1,\dots,k_4 \in \mathbb{Z}_+$, by using Theorem 2. For instance, the equation (31) results from (25) by setting $f_1 = \beta - p$ and $f_2 = -V$ so that f in (14) is given by (11). Furthermore, (29) implies $g = l_1^{k_1}$ (see (13)) so that the Positivstellensatz leads to $s_1 + s_2 f_1 + s_3 f_2 + s_4 f_1 f_2 + g^2 = 0$ which gives (31).
One problem in solving the optimization task (30)–(34) by using SOSTOOLS is the appearance of nonlinear terms of the unknown polynomials q and r in (33)–(34), because only affine linear unknowns can be handled. To circumvent this problem the exponents are set to $k_i = 1$, $\forall i = 3,4$, and the SOS-polynomials s_9, s_{10}, s_{13} and s_{14} are set to zero, thus $(q - u_{up}(r_0 + r))$ in (33) and $(-q + u_{lo}(r_0 + r))$ in (34) can be factored out. As suggested in [4] to keep the degree of the optimization problem low, one chooses $l_i \in \Sigma[x]$ and $s_1,\dots,s_4$ is replaced by $s_1 l_1,\dots,s_4 l_1$, as well as $s_5,\dots,s_8$ with $s_5 l_2,\dots,s_8 l_2$, furthermore the exponents of l_i are set to $k_i = 1$, $\forall i = 1,2$. Afterwards, in the conditions (31)–(32) the polynomial l_i, $\forall i = 1,2$, can be factored out. By solving the result for the SOS-polynomials s_1, s_5, s_{11} and s_{15} the new ***optimization problem 1***

$$\max \beta \qquad \text{s.t.} \tag{35}$$
$$-(\beta - p)s_2 + V s_3 + V(\beta - p)s_4 - l_1 \in \Sigma[x] \tag{36}$$
$$-(\beta - p)s_6 - \frac{\partial V}{\partial x}(f(r_0 + r) - gq)s_7 - \frac{\partial V}{\partial x}(f(r_0 + r) - gq)(\beta - p)s_8 - l_2 \in \Sigma[x] \tag{37}$$
$$-(\beta - p)s_{12} - (q - u_{up}(r_0 + r)) \in \Sigma[x] \tag{38}$$
$$-(\beta - p)s_{16} - (-q + u_{lo}(r_0 + r)) \in \Sigma[x] \tag{39}$$

is obtained. In the end, the constraints (36)–(39) of the optimization problem are described by polynomials which have to be SOS-polynomials. This task can be solved by using a bisection algorithm for searching the upper bound β_{max}. The toolbox SOSTOOLS can be used to check the constraints (36)–(39) by looking for polynomials $V,\ q \in \mathbb{R}[x]$ and $r,\ s_i \in \Sigma[x]$ such that (36)–(39) are fulfilled. As already mentioned, SOSTOOLS can only solve such problems if the unknown polynomials V, q, r and s_i appear affine linear. But in the constraints (36)–(37) they appear bilinear, e.g. $V s_4$, and trilinear, e.g. $\frac{\partial V}{\partial x}(f(r_0 + r) - gq)s_7$. The problem of bi-/trilinearity is solved by an iterative procedure by the following five steps:

1. Step: V, q and r are given and β_1 (solution for β in Step 1) and s_i of (36)–(39) are computed by using a bisection.
2. Step: V, s_7 and s_8 are given and β_2 (solution for β in Step 2), q, r, s_2, s_6, s_{12} and s_{16} in (36)–(39) are computed by using a bisection.
3. Step: V, q and r are given and β_3 (solution for β in Step 3) and s_i of (36)–(39) are computed by using a bisection. (This is a rerun of Step 1 for new q and r)
4. Step: q, r, s_3, s_4, s_7 and s_8 are given and β_4 (solution for β in Step 4), V, s_2, s_6, s_{12} and s_{16} in (36)–(39) are computed by using a bisection.
5. Step: If the difference of the current and the last value of $\beta_{max,4}$ is greater than a chosen tolerance, i.e. $\beta_{max,4,cur} - \beta_{max,4,old} > tol$, go back to Step 1 by using the current functions of the control law q and r and the Lyapunov function V. Otherwise, i.e. $\beta_{max,4,cur} - \beta_{max,4,old} < tol$, stop the iteration and $\beta_{max,4}$ is the solution of the optimization task (35)–(39).

Applying these steps to the optimization problem the unknown polynomials in (36)–(39) appear linear in each step, thus the SOS-programming method can be used. A slight difference to the algorithm in [4] is the additional Step 3 which is a rerun of Step 1 for the new control law q and r computed in Step 2. This additional step improves the solvability of the optimization problem (35)–(39) tremendously, because adequate SOS-polynomials s_7 and s_8 can be computed which have to be fixed in Step 4.
The control law functions q and r for the initialization of the algorithm are a linear state feedback for the linearization of the system (1), e.g. via eigenvalue assignment. The function V for the initialization of the algorithm is the quadratic Lyapunov function of the linearized closed loop system using the initializing linear state feedback. This procedure yields good initial functions and can easily be evaluated. In general no result is known how to choose the degrees of V, q, r and the SOS-polynomials s_i to obtain a solution of the optimization problem. However, one can derive the following restrictions

$$\max\{\deg(ps_2), \deg(Vs_3)\} \geq \max\{\deg(Vps_4), \deg(l_1)\} \tag{40}$$
$$\deg(ps_6) \geq \max\left\{\deg\left(\frac{\partial V}{\partial x}(f(r_0 + r) - gq)s_7\right), \deg\left(\frac{\partial V}{\partial x}(f(r_0 + r) - gq)ps_8\right), \deg(l_2)\right\} \tag{41}$$
$$\deg(ps_{12}) \geq \deg(q - u_{up}(r_0 + r)) \tag{42}$$
$$\deg(ps_{16}) \geq \deg(-q + u_{lo}(r_0 + r)) \tag{43}$$

to ensure that the highest degree in the constraints (36)–(39) is even and has a positive sign which is true for all positive definite polynomials. In the end, a β_{max} is determined together with a control law consisting of q and r (see (3)) and a Lyapunov function $V(x)$ which proves the stability of the equilibrium point $x = 0$ of the closed loop system (4) based on Theorem 1 and considering the input saturation (2). Moreover, an estimation of the region of attraction of the equilibrium point is the second interesting property after providing the stability itself. With the solution β_{max} of the optimization problem (35)–(39) an upper bound for an estimation of the region of attraction on the basis of the obtained result is given by the domain D in (15) in which all necessary conditions for proving asymptotic stability and considering the input saturation (2) are fulfilled.

Estimation of the region of attraction With calculating an upper bound β_{max} and consequently obtaining a domain D where the conditions of Theorem 1 and the restriction of the input saturation (2) are fulfilled the last step of the analysis approach is to get an estimation of the region of attraction of the equilibrium point of the closed loop system (4). To achieve this the largest level set of the Lyapunov function $V(x)$ contained in the domain D is the estimation of the region of attraction (see e.g. [5]). This can be formulated as an optimization task

$$\max c \qquad \text{s.t.} \tag{44}$$
$$\{x \in \mathbb{R}^n \,|V(x) \leq c\} \subseteq \{x \in \mathbb{R}^n \,|p(x) \leq \beta_{max}\} \tag{45}$$

with the upper bound β_{max} and the Lyapunov function $V(x)$ determined by solving (35)–(39). To solve this optimization problem (44)–(45) by using a SOS-program a reformulation is necessary. Therefore, the constraint (45) has to be formulated as an empty set, which leads to the optimization problem

$$\max c \qquad \text{s.t.} \tag{46}$$
$$\{x \in \mathbb{R}^n \,|c - V(x) \geq 0, p(x) - \beta_{max} \geq 0, p(x) - \beta_{max} \neq 0\} = \emptyset \tag{47}$$

Applying Theorem 2 to (47) yields

$$\max c \qquad \text{s.t.} \tag{48}$$
$$s_{c1} + (c - V)\,s_{c2} + (p - \beta_{max})\,s_{c3} + (c - V)\,(p - \beta_{max})\,s_{c4} + (p - \beta_{max})^{2k} = 0 \tag{49}$$

with $s_{c1}, \ldots, s_{c4} \in \Sigma[x]$ and $k \in \mathbb{Z}_+$. To simplify the problem set $k = 1$ and solve for $s_{c1} \in \Sigma[x]$ giving the optimization problem

$$\max c \qquad \text{s.t.} \tag{50}$$
$$-(c - V)\,s_{c2} - (p - \beta_{max})\,s_{c3} - (c - V)\,(p - \beta_{max})\,s_{c4} - (p - \beta_{max})^2 \in \Sigma[x] \tag{51}$$

which can be solved with a bisection of the parameter c by checking the constraint (51) with SOSTOOLS in each step. This optimization problem (50)–(51) is easier to solve because only the SOS-polynomials s_{c2}, s_{c3} and s_{c4} are unknown, thus not iteration is necessary. Again the degrees of the SOS-polynomials have to be chosen which is not an easy task, but there is the restriction

$$\max\{\deg(Vs_{c2}),\ \deg(Vps_{c4})\} \geq \max\left\{\deg(ps_{c3}),\ \deg\left(p^2\right)\right\} \tag{52}$$

which has to be met to ensure a solution of the constraint (51) exists. In the end, the domain $\{x \in \mathbb{R}^n \,|V(x) \leq c_{max}\}$, with c_{max} denoting the solution of (50)–(51), describes the estimation of the region of attraction of the equilibrium point $x = 0$ of the closed loop system (4) with considering the input saturation (2).

4.2 Optimization of the lower bound

The second algorithm is an extension of the so-called "expanding interior algorithm" in [4]. The basic idea of this procedure is to optimize a domain containing the equilibrium point which lies entirely within the region of attraction. Thus the estimation of the region of attraction of the equilibrium point enlarges as the optimization domain is maximized. First of all a polynomial description of the optimization domain has to be defined by

$$P := \{x \in \mathbb{R}^n \,|p(x) \leq \beta\} \tag{53}$$

with a positive definite polynomial $p(x) > 0$, $\forall x \in \mathbb{R}^n \setminus \{0\}$, and $\beta \geq 0$ to ensure that P is connected and contains the equilibrium point $x = 0$. Different from the optimization of the upper bound the largest level set of the unknown Lyapunov function is fixed so the estimation of the region of attraction following this method is given by

$$D := \{x \in \mathbb{R}^n \,|V(x) \leq 1\} \tag{54}$$

To provide the stability of the origin the conditions (5)–(6) of Theorem 1 and the conditions (7)–(8) must be met on the estimation (54) of the region of attraction of the equilibrium point and the optimization domain P (see (53)) must lie within D, which yields

$$\mathbb{R}^n \setminus \{0\} = \{x \in \mathbb{R}^n \,|V(x) > 0\} \tag{55}$$
$$P \subseteq \{x \in \mathbb{R}^n \,|V(x) < 1\} \tag{56}$$
$$D \setminus \{0\} \subseteq \left\{x \in \mathbb{R}^n \,\middle|\, \frac{\partial V(x)}{\partial x}\,(f(x)\,(r_0 + r(x)) - g(x)q(x)) < 0\right\} \tag{57}$$
$$D \subseteq \{x \in \mathbb{R}^n \,|q(x) \leq u_{up}\,(r_0 + r(x))\} \tag{58}$$
$$D \subseteq \{x \in \mathbb{R}^n \,|q(x) \geq u_{lo}\,(r_0 + r(x))\} \tag{59}$$

As already mentioned in the optimization of the upper bound procedure, all unknown functions in (55)–(59) have to be polynomial, except the denominator of the control law $r(x)$ (see (3)) has to be a SOS-polynomial to avoid singularities in the input signal. The condition of a positive definite Lyapunov function in the whole state space (see (55)) is not really a stronger restriction compared to (16) because it turned out that all resulting Lyapunov functions for any example using the upper bound method were positive definite in the whole state space.
In order to achieve a good estimation of the region of attraction of the equilibrium point, the lower bound P has to be maximized and since D must contain P (see (56)) the estimation of the region of attraction is enlarged. Searching for the maximum of β subject to the constraints (55)–(59) shall be solved by using the SOS-programming method, therefore the conditions (55)–(59) have to be reformulated into polynomials which must be checked for a SOS-representation. This can be achieved by using the Positivstellensatz (see Theorem 2), when writing the optimization problem with the constraints (55)–(59) in the form

$$\max \beta \qquad \text{s.t.} \tag{60}$$

$$\{x \in \mathbb{R}^n \,|\, -V(x) \geq 0,\, x \neq 0\} = \emptyset \tag{61}$$

$$\{x \in \mathbb{R}^n \,|\, \beta - p(x) \geq 0,\, V(x) - 1 \geq 0,\, V(x) - 1 \neq 0\} = \emptyset \tag{62}$$

$$\left\{x \in \mathbb{R}^n \,\middle|\, 1 - V(x) \geq 0,\, x \neq 0,\, \frac{\partial V(x)}{\partial x}\left(f(x)(r_0 + r(x)) - g(x)q(x)\right) \geq 0\right\} = \emptyset \tag{63}$$

$$\left\{x \in \mathbb{R}^n \,\middle|\, 1 - V(x) \geq 0,\, q(x) - u_{up}(r_0 + r(x)) \geq 0,\, q(x) - u_{up}(r_0 + r(x)) \neq 0\right\} = \emptyset \tag{64}$$

$$\left\{x \in \mathbb{R}^n \,\middle|\, 1 - V(x) \geq 0,\, -q(x) + u_{lo}(r_0 + r(x)) \geq 0,\, -q(x) + u_{lo}(r_0 + r(x)) \neq 0\right\} = \emptyset \tag{65}$$

The non-polynomial inequalities $x \neq 0$ in (61) and (63) have to be replaced by polynomial inequalities using the positive definite polynomials $l_i(x) > 0$, $\forall x \in \mathbb{R}^n \setminus \{0\}$ and $i = 1,2$ (see (29)). Using Theorem 2 leads to the optimization problem

$$\max \beta \qquad \text{s.t.} \tag{66}$$

$$s_1 - Vs_2 + l_1^{2k_1} = 0 \tag{67}$$

$$s_3 + (\beta - p)s_4 + (V-1)s_5 + (\beta - p)(V-1)s_6 + (V-1)^{2k_2} = 0 \tag{68}$$

$$s_7 + (1-V)s_8 + \frac{\partial V}{\partial x}(f(r_0 + r) - gq)s_9 + \frac{\partial V}{\partial x}(f(r_0 + r) - gq)(1-V)s_{10} + l_2^{2k_3} = 0 \tag{69}$$

$$s_{11} + (1-V)s_{12} + (q - u_{up}(r_0 + r))s_{13} + (q - u_{up}(r_0 + r)(1-V)s_{14} + (q - u_{up}(r_0 + r))^{2k_4} = 0 \tag{70}$$

$$s_{15} + (1-V)s_{16} + (-q + u_{lo}(r_0 + r))s_{17} + (-q + u_{lo}(r_0 + r))(1-V)s_{18} + (-q + u_{lo}(r_0 + r))^{2k_5} = 0 \tag{71}$$

with $s_1, \ldots, s_{18} \in \Sigma[x]$ and $k_1, \ldots, k_5 \in \mathbb{Z}_+$. Again, the problem of nonlinear terms of the unknown polynomials q, r and in addition V in (68)–(71) must be solved by setting the exponents to $k_i = 1$, $\forall i = 2,4,5$, and the SOS-polynomials s_3, s_4, s_{10}, s_{11}, s_{12}, s_{15} and s_{16} are set to zero, thus $V-1$ in (68), $(q - u_{up}(r_0 + r))$ in (70) and $(-q + u_{lo}(r_0 + r))$ in (71) can be factored out and the quadratic term of the Lyapunov function in (69) is removed. To keep the degree of the optimization problem low, one chooses $l_i \in \Sigma[x]$ and s_1 is replaced by $s_1 l_1$, s_2 by l_1, as well as $s_7, \ldots, s_9$ with $s_7 l_2, \ldots, s_9 l_2$, furthermore the exponents of l_i are set to $k_i = 1$, $\forall i = 1,3$. Afterwards, in the conditions (67) and (69) the polynomial l_i, $\forall i = 1,2$, can be factored out. By solving the result for the SOS-polynomials s_1, s_5, s_7, s_{13} and s_{17} the new ***optimization problem 2***

$$\max \beta \qquad \text{s.t.} \tag{72}$$

$$V - l_1 \in \Sigma[x] \tag{73}$$

$$-(\beta - p)s_6 - (V-1) \in \Sigma[x] \tag{74}$$

$$-(1-V)s_8 - \frac{\partial V}{\partial x}(f(r_0 + r) - gq)s_9 - l_2 \in \Sigma[x] \tag{75}$$

$$-(1-V)s_{14} - (q - u_{up}(r_0 + r)) \in \Sigma[x] \tag{76}$$

$$-(1-V)s_{18} - (-q + u_{lo}(r_0 + r)) \in \Sigma[x] \tag{77}$$

is obtained. Finally, the polynomial constraints (73)–(77) are derived which have to be checked for a SOS representation, but once again there appear unknown polynomials bi-/trilinear in the constraints (75)–(77), e.g. $(1-V)s_{14}$ or $\frac{\partial V}{\partial x}(f(r_0 + r) - gq)s_9$. This problem of bi-/trilinear appearance of unknown polynomials is solved in a similar way compared to the upper bound procedure with an iterative algorithm. Remembering the iterative procedure in Section 4.1, the basic idea is to fix one or two unknown polynomials (V, q, r and s_i) such that the remaining unknowns only appear affine linear and consequently the SOS-program can be solved. By having a closer look on the constraint (73), the only unknown polynomial is the Lyapunov function V which will be fixed for certain steps of the iterative procedure, thus the condition (73) can be omitted in all cases where V is fixed.

The constraint (74) represents the condition that the optimization domain P (see (53)) must lie within the estimation of the region of attraction (see (56)). Hence, when V is fixed (74) always results in the same β solving the optimization problem (72)–(77). As the only other unknown polynomial s_6 in (74) only appears affine linear it doesn't need to be fixed but can be calculated in each step. Consequently, the constraint (74) can be omitted in all cases of the fixed Lyapunov function V. This results in an less complex optimization problem however by omitting (73)–(74) the optimization parameter β gets lost. By introducing the new optimization parameter α in $V \leq \alpha$ compared to $V \leq 1$ (see (54)) the new ***optimization problem 3***

$$\max \alpha \qquad \text{s.t.} \tag{78}$$

$$-(\alpha - V)s_8 - \frac{\partial V}{\partial x}(f(r_0 + r) - gq)s_9 - l_2 \in \Sigma[x] \tag{79}$$

$$-(\alpha - V)s_{14} - (q - u_{up}(r_0 + r)) \in \Sigma[x] \tag{80}$$

$$-(\alpha - V)s_{18} - (-q + u_{lo}(r_0 + r)) \in \Sigma[x] \tag{81}$$

is derived which maximizes the estimation of the region of attraction in all iteration steps with a fixed Lyapunov function V. The optimization problem 3 (see (78)–(81)) has the advantage that an easier choice of the initial control law q and r and the initial Lyapunov function V arises where according to the optimization of the upper bound a linear state feedback for the linearization of system (1) and the quadratic Lyapunov function of the linearized closed loop system is used. Without omitting the conditions (73)–(74) the initial functions V, q and r would have to be chosen such that the estimation of the region of attraction, i.e. $V \leq 1$, lies within the domain of the input saturation (see (58)–(59)). Using the modified optimization problem 3 for fixed V (78)–(81) the estimation of the region of attraction, now given by $V \leq \alpha$, is determined by optimizing α such that (58)–(59) hold automatically. In other words with the modified optimization problem 3 the estimation of the region of attraction of the equilibrium point is computed for the initial V, q and r such that the input saturation is fulfilled instead of a difficult predefinition of the initial V, q and r which satisfies the input saturation a priori. In the end, the iterative procedure for the avoidance of the bi-/trilinearities of the unknown polynomials consisting of five steps is given by:

1. Step: Solving optimization problem 2 (78)–(81) with given V, q and r such that α_1 (solution for α in Step 1) and s_i of (79)–(81) are computed by using a bisection.
2. Step: Solving optimization problem 2 (78)–(81) with given V and s_9 such that α_2 (solution for α in Step 2), q, r, s_8, s_{14} and s_{18} of (79)–(81) are computed by using a bisection.
3. Step: Solving optimization problem 2 (78)–(81) with given V, q and r such that α_3 (solution for α in Step 3) and s_i of (79)–(81) are computed by using a bisection. (This is a rerun of Step 1 for new q and r)
4. Step: Solving optimization problem 3 (72)–(77) with given q, r, s_8, s_9, s_{14} and s_{18} such that β_4 (solution for β in Step 4), V and s_6 in (73)–(77) are computed by using a bisection.
5. Step: If the difference of the current and the last value of $\beta_{max,4}$ is greater than a chosen tolerance, i.e. $\beta_{max,4,cur} - \beta_{max,4,old} > tol$, go back to Step 1 by using the current functions of the control law q and r and the Lyapunov function V. Otherwise, i.e. $\beta_{max,4,cur} - \beta_{max,4,old} < tol$, stop the iteration and $\beta_{max,4}$ is the solution of the optimization task (72)–(77).

Applying these steps to the optimization problem the unknown polynomials in (73)–(77) and (79)–(81) respectively remain linear in each step, thus the SOS-programming method can be used. In the end, the optimization problem 2 is only used in Step 4 where a new Lyapunov function V is calculated, in Step 1 to 3 the less complex optimization problem 3 for a fixed Lyapunov function V can be used which reduces the computational effort slightly.
In general no result is known how to choose the degrees of V, q, r and the SOS-polynomials s_i to obtain a solution of the optimization problem. However, one can derive the following restrictions

$$\deg(V) \geq \deg(l_1) \tag{82}$$

$$\deg(ps_6) \geq \deg(V) \tag{83}$$

$$\deg(Vs_8) \geq \max\left\{\deg\left(\frac{\partial V}{\partial x}(f(r_0 + r) - gq)s_9\right),\ \deg(l_2)\right\} \tag{84}$$

$$\deg(Vs_{14}) \geq \deg(q - u_{up}(r_0 + r)) \tag{85}$$

$$\deg(Vs_{18}) \geq \deg(-q + u_{lo}(r_0 + r)) \tag{86}$$

to ensure that the conditions (73)–(77) and (79)–(81) respectively can be met. In the end, a β_{max} is determined together with a control law consisting of q and r (see (3)) and a Lyapunov function $V(x)$ which proves the stability of the equilibrium point $x = 0$ of the closed loop system (4) based on Theorem 1 and considering the input saturation (2). Moreover, the estimation of the region of attraction of the equilibrium point $x = 0$ of the closed loop system (4) using the procedure of the optimization of the lower bound is directly given by the domain $\{x \in \mathbb{R}^n \,|\, V(x) \leq 1\}$.

5 Example

In this section the presented approach is used to construct a rational control law (3) and a Lyapunov function to prove the stability and estimate the region of attraction of the equilibrium point of the closed loop system (4). The system is given by the polynomial description

$$\dot{x} = \begin{bmatrix} -x_2 \\ x_1 + x_2\left(x_1^2 - 1\right) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad -1 \leq u \leq 1 \tag{87}$$

with an equilibrium point at $x = 0$ for $u = 0$ which will be stabilized with the aim of an as large as possible estimation of the region of attraction and the consideration of the input saturation as shown in the presented design approach.

First of all, a stabilizing control law and a Lyapunov function is constructed for (87) by using the upper bound procedure, see Section 4.1. Therefore, the domain D which is the upper bound for the estimation of the region of attraction must be specified by choosing a positive definite polynomial for $p(x)$ (see (15)), thus the shape of the upper bound has to be predefined. It turned out in many examples that quadratic polynomials lead to good results, because the overall SOS-programs (35)–(39) and (50)–(51) become as simple as possible and, thus, easier to solve. For the considered example (87) which are the known van der pol equations (see [9]) extended with an input, hence the same ellipse as in [9] is chosen for $p(x)$, i.e. $p(x) = 0.378x_1^2 - 0.274x_1x_2 + 0.278x_2^2$. For the reason of low complexity all positive definite functions l_i are chosen to $l_i(x) = 10^{-4}(x_1^2 + x_2^2)$.

As already mentioned in Section 4.1, for the first calculation step of β in (35)–(39) a control law $q(x)$, $r(x)$ and a function $V(x)$ must be provided. Therefore, a linear state feedback for the Jacobian linearization of (87) is computed with the eigenvalue assignment of $\lambda_1 = -2$ and $\lambda_2 = -3$. Afterwards, the quadratic Lyapunov function $V_{lin}(x)$ of the Jacobian linearization of the closed loop system using the linear state feedback is chosen for the initial function of $V(x)$.

The last step before solving (35)–(39) and (50)–(51) is to ascertain the degrees of all unknown polynomials. In an first attempt a linear control law is chosen, i.e. $\deg(q) = 1$, $\deg(r) = 0$, and the constant r_0 in the denominator (see (3)) is set to $r_0 = 0.1$ for all further considerations. Furthermore, a quadratic Lyapunov function is picked together with the degrees of the unknown SOS-polynomials s_i in (35)–(39) and s_{ci} in (50)–(51) which are set to $\deg(s_{2,3}) = \deg(s_{12}) = \deg(s_{16}) = 2$, $\deg(s_4) = \deg(s_{7,8}) = 0$ and $\deg(s_6) = \deg(s_{ci}) = 4$. Using the stated settings Figure 1 shows the estimation of the region of attraction of the closed loop system and the limits of the input saturation of system (87). Clearly, a linear control law and a Lyapunov function is constructed which proves

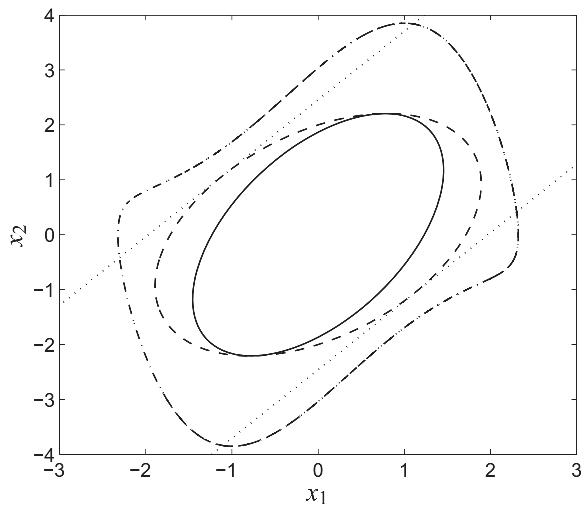

Figure 1: Estimation of the region of attraction (RoA) for deg(V)=2, deg(q)=1 and deg(r)=0: (-·-·) RoA of the closed loop system, (···) input saturation, (– –) $p = 1.1118$, (—) estimation of the RoA of the closed loop system ($V = 0.4946$)

asymptotic stability of the equilibrium point $x = 0$ of the closed loop system and an estimation of the region of attraction is evaluated with $\beta_{max} = 1.1118$ by using (35)–(39) and $c_{max} = 0.4946$ by using (50)–(51). Furthermore, the upper bound in Figure 1 lies entirely within the input saturation so conditions (18)–(19) are fulfilled. In order to see how the dynamics of the open loop system (87) are changed by the calculated control law, the eigenvalues of the linearizations are compared. The open loop system (87) has eigenvalues at $\lambda_{1,2} = -0.5 \pm j0.86$ and the eigenvalues of the closed loop systems using the linear control law are $\lambda_{1,2} = -0.70 \pm j1.01$. So the dynamics of the closed loop system are slightly faster in this special case but one has to keep in mind, there is no influence on the resulting dynamics of the closed loop systems using the presented design method. The exact region of attraction of the equilibrium point of the closed loop system of (87) is shown in Figure 1 which is an unstable limit cycle and can be identified by a simulation in reverse time, for details see [5]. Comparing the estimation with the exact region of attraction of the equilibrium point reveals a large difference.

The result for a nonlinear control law, i.e. $\deg(q) = 5$ and $\deg(r) = 2$, a Lyapunov function of degree 6 and the setting $\deg(s_{2,3}) = 6$, $\deg(s_6) = 10$, $\deg(s_4) = \deg(s_{7,8}) = 0$, $\deg(s_{12}) = \deg(s_{16}) = 4$ and $\deg(s_{ci}) = 4$ is shown in Figure 2. This setting yields $\beta_{max} = 2.2587$ and $c_{max} = 5.178$ which is a huge improvement compared to the

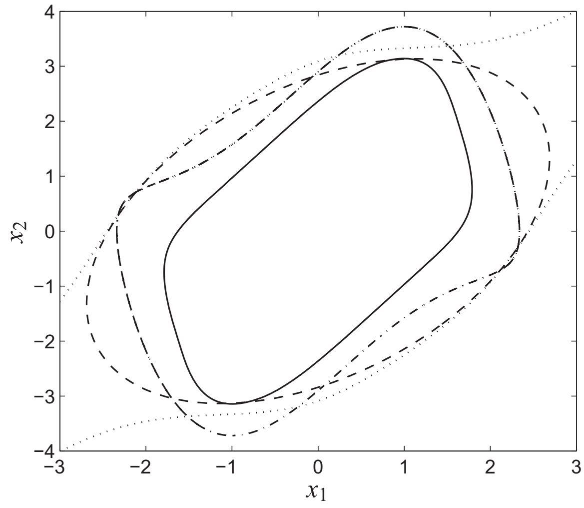

Figure 2: Estimation of the region of attraction (RoA) for deg(V)=6, deg(q)=5 and deg(r)=2: (-·-·) RoA of the closed loop system, $(\cdots)$ input saturation, (– –) $p = 2.2587$, (—) estimation of the RoA of the closed loop system ($V = 5.178$)

linear control law and the quadratic Lyapunov function in Figure 1. Clearly, the nonlinear control law yields a more flexible shape of the input saturation which yields a larger upper bound β_{max}, furthermore the Lyapunov function of degree 6 is also more flexible, thus it can adopt the shape of the exact region of attraction of the equilibrium point better (see Figure 2). The resulting eigenvalues of the linearization are $\lambda_{1,2} = -0.82 \pm j0.57$ which are slightly faster than the eigenvalues of the open loop system and the closed loop system using a linear control law.

Second, the optimization of the lower bound (see Section 4.2) is used for the construction of a control law and a Lyapunov function. Again, the positive definite polynomial $p(x)$ has to be defined for the optimization region P (see (53)). In order to keep the optimization task as simple as possible a quadratic function is chosen, although it is uninspired a circle will yield good results, i.e. $p(x) = x_1^2 + x_2^2$, and the positive definite functions l_i are set to $l_i(x) = 10^{-4}(x_1^2 + x_2^2)$ like in the case of the optimization of the upper bound. Because the open loop system (87) is stable, the initial control function is set to zero, i.e. $r(x) = 0$. The initial function $V(x)$ is the quadratic Lyapunov function $V_{lin}(x)$ of the Jacobian linearization of the open loop system (87). This quadratic Lyapunov function is multiplied by a factor of 0.1 for the consideration of the lower bound method, i.e. $0.1V_{lin}(x)$. It turned out that this factor yields a tremendous improvement of the region of attraction of the equilibrium point. A possible reason for this effect is that the optimization problem (66)–(71) is nonlinear and non convex, hence the solution depends on the initial value because not necessarily a global optimum is achieved. However, the presented design leads to promising results as shown in the further discussion of the example.

Before solving the optimization task (66)–(71) the degrees of all unknown polynomials have to be chosen. In Figure 3 the result is shown for a nonlinear control law with $\deg(q) = 3$ and $\deg(r) = 2$, a Lyapunov function of degree 4 and the setting $\deg(s_6) = \deg(s_{14}) = \deg(s_{18}) = 4$, $\deg(s_8) = 6$ and $\deg(s_9) = 0$ for the unknown polynomials s_i. The estimation of the region of attraction of the equilibrium point $x = 0$ using the lower bound method is even

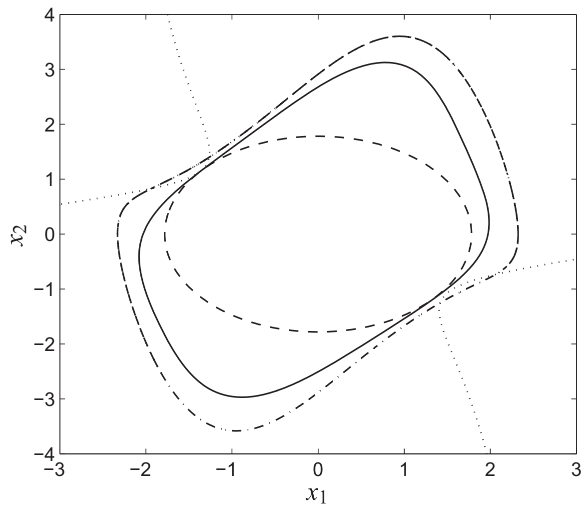

Figure 3: Estimation of the region of attraction (RoA) for deg(V)=4, deg(q)=3 and deg(r)=2: (-·-·) RoA of the closed loop system, $(\cdots)$ input saturation, (– –) $p = 3.1765$, (—) estimation of the RoA of the closed loop system ($V = 1$)

larger than the estimation following the upper bound method (see Figure 2 and Figure 3). Furthermore, as specified in the constraint (56) the lower bound lies entirely within the estimation of the region of attraction $V(x) \leq 1$ (see (54)) and the estimation of the region of attraction lies within the boundaries of the input saturation fulfilling constraints (58)–(59). Using the optimization of the lower bound yields the eigenvalues $\lambda_1 = -0.32$ and -2.36 which are real valued and compared to the open loop system and the closed loop system using the optimization of the upper bound slightly slower. But as already mentioned, there is no influence on the resulting dynamics using the presented approach.
Both procedures reveal advantages compared to a linear state feedback and a quadratic Lyapunov function. The rational control law (3) used in the design methods yields a more flexible shape of the input saturation which becomes quite clear in Figure 2. Furthermore, Lyapunov functions of degree higher than 2 adopt the shape of the region of attraction much better than a quadratic Lyapunov function which is fixed to an ellipsoidal shape.

6 Conclusion

A design method for the stabilization of an equilibrium point of a nonlinear system was presented in this paper. Of main interest analyzing a nonlinear system is the region of attraction of the equilibrium point, therefore a Lyapunov function is needed to get an estimation of the region of attraction. This can be handled by using the SOS-programming method where a stabilizing control law and a suitable Lyapunov function is constructed within an optimization task. Furthermore, the consideration of the input saturation which has tremendous influence on the region of attraction of the equilibrium point is systematically included in the whole design procedure which yields two additional constraints in the SOS-programs. The degrees of freedom of the nonlinear control law are used for the construction of the Lyapunov function and the consideration of the input saturation with the aim of a maximized estimation of the region of attraction of the equilibrium point of the closed loop system. As the SOS-programming technique is restricted to polynomials only systems with polynomial nonlinearities can be considered. Nevertheless, as shown in this paper the nonlinear control law is extended to a rational form consisting of a polynomial nominator and denominator which yields after a small rearrangement polynomial constraints in the SOS-programs. The problem of bi-/trilinearities in the constraints of the SOS-program can be circumvent by an iterative algorithm using a bisection in each step for solving the optimization problem. Following the presented approach, no influence on the dynamics of the closed loop system is possible, but especially instable equilibrium points of the open loop system are guaranteed to be stabilized by the rational control law. In the end, a numerical design procedure for a systematic construction of a nonlinear control law and a polynomial Lyapunov function of degree larger than 2 is derived which often results in a tremendous improvement of the estimation of the region of attraction of the equilibrium point of the closed loop system compared to a linear control law and the standard quadratic Lyapunov function. The basic idea is implemented in two different algorithms where both provided good results for the considered example.
The presented design method can be used for systems with limited state variables (for details see [3]) with a slight modification of the constraints in the SOS-programs. The conditions for the input saturation have to be replaced with one condition for each limit of a state variable which yields possibly more constraints but these can be computed easier because no bilinear terms appear in these constraints. Of course, a combination of input saturation and limited state variables using the SOS-programming technique is also possible.

7 References

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