On a characterization of physical systems by spaces of numerical events

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Abstract. The probability $p(s)$ of the occurrence of an event pertaining to a physical system $\mathbf{S}$ which is observed in different states s determines a function p from the set S of all states of $\mathbf{S}$ to $[0,1]$. The function p is called a numerical event. Sets of numerical events are structured and characterized in different ways. Moreover, properties of the resulting spaces are specified in order to distinguish between quantum mechanical phenomena and measurements within classical physical systems, as well for finite sets S of states as for infinite ones.

1 Introduction

Let S denote the set of all states of a physical system $\mathbf{S}$. A property of $\mathbf{S}$ can be characterized by the family $(p(s); s \in S)$ of probabilities meeting this property. For example one can think of the property of finding the value of some given observable inside a given set of reals. In this case it is even of interest that p may only assume the two values 0 and 1. However, in general, $p(s) \in [0,1] \subseteq \mathbf{R}$ for every $s \in S$. In the following we call the function p a *numerical event*. Also the notion *multidimensional probability* is sometimes used (cf. [2] and [3]).

Let P be a subset of the set $[0,1]^S$ of all functions from S to $[0,1]$. Then P is ordered in a natural way by the partial order of functions, i. e. $p \leq q$ if $p(s) \leq q(s)$ for all $s \in S$. It is also obvious to assume that P has the following properties:

(1) $0 \in P$

(2) If $p \in P$ then $p' := 1 - p \in P$.

(3) If $p, q \in P$ and $p + q \leq 1$ then $p + q \in P$.

(0 and 1 denote the corresponding constant functions.)
Elements p, q of P such that $p + q \leq 1$ are called *orthogonal*, which we will denote by $p \perp q$.
If, in addition to (1) – (3) P fulfils

(4) If $p, q, r \in P$ and $p \perp q \perp r \perp p$ then $p + q + r \leq 1$.

then P is called an *algebra of S-probabilities* or, less precisely, an *algebra of numerical events*. (cf. [2] and [3]; in these papers both conditions (3) and (4) are substituted by the one condition

(5) If $p, q, r \in P$ and $p \perp q \perp r \perp p$ then $p + q + r \in P$.

which leads to an equivalent definition.)
Three elements p, q, r of P such that $p \perp q \perp r \perp p$ will be called an *orthogonal triple* (p, q, r) of $(P, \leq, ')$.

Algebras of S-probabilities have been studied from the purely algebraic point of view concerning only the structure induced by the order $\leq$ in [6], then, in respect to applications in quantum mechanics, in [2], [3] and recently in [1]. Algebras of S-probabilities have the property that $p \perp q$ implies that $p + q$ is the supremum $p \vee q$ of p and q in $(P, \leq)$. Moreover, if for arbitrary pairs (p, q) of elements of P the supremum $p \vee q$ exists in $(P, \leq)$ or, as one can show to be equivalent, if for every pair (p, q) of elements of P the infimum $p \wedge q$ exists in $(P, \leq)$, then $(P, \leq, ')$ is a so-called orthomodular lattice. If an orthomodular

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lattice is distributive, it is a Boolean algebra. (For definitions and properties of lattices confer any book on lattice theory.)
As pointed out in [2] and [3], algebras of S-probabilities can serve as (generalized) event fields for quantum mechanical models and, moreover, a classical physical system can be characterized by the property that the algebra of S-probabilities is a Boolean algebra, which is familiar to us from event fields in traditional probability theory.

Example 1.1. $\{0, x, 1-x, |2x-1|, 1-|2x-1|, 1\}$ is an algebra of $[0,1]$-probabilities. (Observe that here $x \vee |2x-1| = 1 \neq \max(x, |2x-1|)$.)

Example 1.2. For every positive integer n, $\{0,1\} \cup \{(\pi-1)/(n-1) \mid \pi \in S_n\}$ is an algebra of $\{1,\ldots,n\}$-probabilities. (S_n denotes the set of all permutations of $\{1,\ldots,n\}$.)

Example 1.3. Based on the classical Hilbert space model of quantum mechanics let H be a Hilbert space, S denote the set of all one-dimensional subspaces of H, and for every $s \in S$ let a_s be a fixed unit vector in s. We further denote the set of all projectors of H by $P(H)$ and the inner product in H by $(.,.)$. Then $\{s \mapsto (Qa_s, a_s) \mid Q \in P(H)\}$ is an algebra of S-probabilities (cf. [3]).

In this paper we first identify algebras of S-probabilities among sets of numerical events and specify conditions for these sets to be Boolean algebras. From the obtained results we then derive ways to find out by measurements whether one deals with a quantum mechanical system or a classical one. Recently, this distinction has become very important in connection with the design of computer chips, but, of course, this is of general interest in quantum mechanics. As for the set S of states we assume that S can be small as well as infinitely large.

2 Characterizing algebras of S-probabilities

As usual, we denote the set of all subsets of S by 2^S.

Theorem 2.1. If every $p \in P$ can only assume the values 0 and 1, then P is an algebra of S-probabilities if and only if $P = \{I_X \mid X \in \mathscr{M}\}$ where I_X denotes the indicator function on X and $\mathscr{M} \subseteq 2^S$ satisfies (i) – (iii):

(i) $\emptyset \in \mathscr{M}$
(ii) If $A \in \mathscr{M}$ then $A' := S \setminus A \in \mathscr{M}$.
(iii) If $A, B \in \mathscr{M}$ and $A \cap B = \emptyset$ then $A \cup B \in \mathscr{M}$.

Moreover, if $A \cup B \in \mathscr{M}$ for all $A, B \in \mathscr{M}$, then $(P, \leq, ')$ is a Boolean algebra.
($\cup$ and $\cap$ denote the join and meet of sets, respectively.)

Proof. Let $\mathscr{M} \subseteq 2^S$ and put $P := \{I_X \mid X \in \mathscr{M}\}$.
First, we assume that P is an algebra of S-probabilities. Then

(i) $I_\emptyset = 0 \in P$
(ii) If $A \in \mathscr{M}$ then $I_A \in P$ whence $I_{A'} = (I_A)' \in P$ from which we infer $A' \in \mathscr{M}$.
(iii) If $A, B \in \mathscr{M}$ and $A \cap B = \emptyset$ then $I_A, I_B \in P$ and $I_A \perp I_B$ which shows $I_{A \cup B} = I_A + I_B \in P$, and hence $A \cup B \in \mathscr{M}$.

Conversely, assume $\mathscr{M}$ to satisfy (i) – (iii) and let $A, B, C \subseteq S$.

(1) $0 = I_\emptyset \in P$
(2) If $I_A \in P$ then $A \in \mathscr{M}$ and therefore $A' \in \mathscr{M}$ which means that $(I_A)' = I_{A'} \in P$.
(3) If $I_A, I_B \in P$ and $I_A \perp I_B$ then $A \cap B = \emptyset$ and hence $A \cup B \in \mathscr{M}$, consequently $I_A + I_B = I_{A\cup B} \in P$.
(4) If $I_A, I_B, I_C \in P$ and $I_A \perp I_B \perp I_C \perp I_A$ then A, B, C are pairwise disjoint and therefore $A \cup B \in \mathscr{M}$. Because of $(A \cup B) \cap C = \emptyset$ this implies $A \cup B \cup C = (A \cup B) \cup C \in \mathscr{M}$ from which we infer $I_A + I_B + I_C = I_{A\cup B\cup C} \leq 1$.

If $A \cup B \in \mathscr{M}$ for all $A, B \in \mathscr{M}$, then $(\mathscr{M}, \subseteq, ')$ is a Boolean algebra wherefrom one can easily derive that $(P, \leq, ')$ is also a Boolean algebra. □

Remark 2.2. $(P, \leq, ')$ is an orthomodular lattice if and only if $(\mathscr{M}, \subseteq)$ is a lattice and $(P, \leq, ')$ is a Boolean algebra if and only if $(\mathscr{M}, \subseteq)$ is a distributive lattice.

Proof. This follows immediately from the representation of P given in Theorem 2.1. □

Theorem 2.3. If every $p \in P$ can assume one of the values 0, $1/2$ and 1 then P is an algebra of S-probabilities if and only if $P = \{I_A + (1/2)I_B \,|\, (A, B) \in \mathscr{M}\}$ where $\mathscr{M}$ is a set of pairs of subsets of S satisfying

(i) $A \cap B = \emptyset$ for all $(A, B) \in \mathscr{M}$
(ii) $(\emptyset, \emptyset) \in \mathscr{M}$
(iii) If $(A, B) \in \mathscr{M}$ then $(A, B)' := (S \setminus (A \cup B), B) \in \mathscr{M}$.
(iv) If $(A, B), (C, D) \in \mathscr{M}$ and $A \cap C = (A \cup C) \cap (B \cup D) = \emptyset$ then $(A \cup C \cup (B \cap D), B \triangle D) \in \mathscr{M}$.
(v) If $(A, B), (C, D), (E, F) \in \mathscr{M}$ and $A \cap C = C \cap E = E \cap A = (A \cup C \cup E) \cap (B \cup D \cup F) = \emptyset$ then $B \cap D \cap F = \emptyset$.

($\triangle$ denotes the symmetric difference of sets.)

Proof. Let $\mathscr{M}$ be a set of pairs of subsets of S and put $P := \{I_X + (1/2)I_Y \,|\, (X, Y) \in \mathscr{M}\}$.
First we assume that P is an algebra of S-probabilities.

(i) If $(A, B) \in \mathscr{M}$ then $p := I_A + (1/2)I_B \in P$ and hence $p \leq 1$ which means $A \cap B = \emptyset$.
(ii) $I_\emptyset + (1/2)I_\emptyset = 0 \in P$
(iii) If $(A, B) \in \mathscr{M}$ then $p := I_A + (1/2)I_B \in P$ and therefore $I_{S\setminus(A\cup B)} + (1/2)I_B = p' \in P$ whence $(A, B)' \in \mathscr{M}$.
(iv) If $(A, B), (C, D) \in \mathscr{M}$ and $A \cap C = (A \cup C) \cap (B \cup D) = \emptyset$ then $p := I_A + (1/2)I_B \in P$, $q := I_C + (1/2)I_D \in P$ and $p \perp q$. Therefore $I_{A\cup C\cup(B\cap D)} + (1/2)I_{B\triangle D} = p + q \in P$ wherefrom it follows that $(A \cup C \cup (B \cap D), B \triangle D) \in \mathscr{M}$.
(v) If $(A, B), (C, D), (E, F) \in \mathscr{M}$ and $A \cap C = C \cap E = E \cap A = (A \cup C \cup E) \cap (B \cup D \cup F) = \emptyset$ then $p := I_A + (1/2)I_B \in P$, $q := I_C + (1/2)I_D \in P$, $r := I_E + (1/2)I_F \in P$ and $p \perp q \perp r \perp p$, which implies $p + q + r \leq 1$, hence $B \cap D \cap F = \emptyset$.

Conversely, assume $\mathscr{M}$ to satisfy (i) – (v) and let $(A, B), (C, D), (E, F)$ be pairs of subsets of S.

(1) $0 = I_\emptyset + (1/2)I_\emptyset \in P$
(2) If $p := I_A + (1/2)I_B \in P$ then $(A, B) \in \mathscr{M}$ and hence $(A, B)' \in \mathscr{M}$, wherefrom we infer $p' = I_{S\setminus(A\cup B)} + (1/2)I_B \in P$.
(3) If $p := I_A + (1/2)I_B \in P$, $q := I_C + (1/2)I_D \in P$ and $p \perp q$ then $(A, B), (C, D) \in \mathscr{M}$ and $A \cap C = (A \cup C) \cap (B \cup D) = \emptyset$ from which we can conclude $(A \cup C \cup (B \cap D), B \triangle C) \in \mathscr{M}$. Therefore $p + q = I_{A\cup C\cup(B\cap D)} + (1/2)I_{B\triangle C} \in P$.

(4) If $p := I_A + (1/2)I_B \in P$, $q := I_C + (1/2)I_D \in P$, $r := I_E + (1/2)I_F \in P$ and $p \perp q \perp r \perp p$ then $A \cap C = C \cap E = E \cap A = (A \cup C \cup E) \cap (B \cup D \cup F) = \emptyset$ and hence $B \cap D \cap F = \emptyset$, which means $p + q + r \leq 1$.

□

By means of Theorem 2.3 one can prove the following

Remark 2.4. If one defines a binary relation $\leq$ on $\mathscr{M}$ by $(A,B) \leq (C,D)$ if $A \subseteq C$ and $B \subseteq C \cup D$ for every $(A,B),(C,D) \in \mathscr{M}$ then $(P,\leq,')$ turns out to be an orthomodular lattice if and only if $(\mathscr{M},\leq)$ is a lattice, and $(P,\leq,')$ will be a Boolean algebra if and only if $(\mathscr{M},\leq)$ is a distributive lattice.

Example 2.5. $\mathscr{M} := \{(\emptyset,\emptyset),(\{1,2,3\},\emptyset)\} \cup \{(\{i\},\{j\}) \,|\, i,j \in \{1,2,3\}, i \neq j\}$ satisfies (i) – (v) for $S = \{1,2,3\}$.

Remark 2.6. As one can show by means of the representation given in Theorems 2.1 and 2.3, for $|S| = 1,2$ resp. 3 there are exactly $1,2$ resp. 5 algebras of S-probabilities $P \subseteq \{0,1\}^S$ and exactly $1,2$ resp. 15 such algebras $P \subseteq \{0,1/2,1\}^S$.

Now we put more structure on P by defining

$$p \oplus q := \begin{cases} p+q & \text{if } p \perp q \\ |p-q| & \text{otherwise} \end{cases}$$

for all $p,q \in P$.

Example 2.7. $\{0,1\}^S$ is an algebra of S-probabilities which is closed with respect to $\oplus$.

Example 2.8. $\{(0,0,0),(1,1,1),(1,0,0),(0,1,1),(1/2,0,1),(1/2,1,0)\}$ is an algebra of $\{s_1,s_2,s_3\}$-probabilities which is closed with respect to $\oplus$.

Example 2.9. If for every $s \in S$ the number n_s is a positive integer then

$$\{p \in [0,1]^S \,|\, p(s) \in \{\frac{0}{n_s},\frac{1}{n_s},\frac{2}{n_s},\ldots,\frac{n_s}{n_s}\} \text{ for all } s \in S\}$$

is a set of numerical events which satisfies (1) – (3) and is closed with respect to $\oplus$.

Our goal is to make sure to deal with an algebra of S-probabilities. The following theorem provides a sufficient condition for this.

Theorem 2.10. If P satisfies (1) – (3), $p(s) > 0$ for all $p \in P \setminus \{0\}$ and all $s \in S$ and there is no $p \in P \setminus \{0\}$ with $p \leq 1/2$ then P is an algebra of S-probabilities if $(p \oplus q) \oplus r = p \oplus (q \oplus r)$ for every orthogonal triple (p,q,r) of $(P,\leq,')$.

Proof. Suppose there exists an orthogonal triple (p,q,r) of $(P,\leq,')$ with $p+q+r \not\leq 1$. Then $|p+q-r| = (p \oplus q) \oplus r = p \oplus (q \oplus r) = |p-q-r|$ from which we infer $(p+q-r)^2 = (p-q-r)^2$ whence $(p-r)q = 0$. Because of $p+q+r \not\leq 1$ and $p \perp r$ we have $q \neq 0$ and hence $q(s) > 0$ for all $s \in S$. This shows $p = r$ which together with $p \perp r$ implies $p = r \leq 1/2$ wherefrom we can conclude $p = r = 0$ contradicting $p+q+r \not\leq 1$. Therefore (4) holds. □

Theorem 2.11. An algebra P of S-probabilities is a Boolean algebra if and only if for all $f,g \in P$ there exist $p,q \in P$ with $f-p \in P$, $p \perp q$ and $f+q = p+g$.

Proof. Let $f,g,p,q \in P$. If $f-p \in P$, $p \perp q$ and $f+q = p+g$ then $(p, f-p, q)$ is an orthogonal triple of $(P, \leq, ')$, $p+(f-p) = f$ and $(f-p)+q = (g-q)+q = g$.
Conversely, if (p,r,q) is an orthogonal triple of $(P, \leq, ')$ such that $p+r = f$ and $r+q = g$ then $f-p = r \in P$, $p \perp q$ and $f+q = (p+r)+q = p+(r+q) = p+g$.
According to a result in [2] an algebra of S-probabilities is a Boolean algebra if and only if for every pair f,g of elements of P there exists an orthogonal triple (p,q,r) of $(P, \leq, ')$ such that $p+q = f$ and $q+r = g$, which completes the proof of the theorem. $\square$

Suggested by the standard Hilbert space model for algebras of S-probabilities which can be shown to be a lattice, we next assume that with $p,q \in P$ also the supremum $p \vee q$ and the infimum $p \wedge q$ belong to P. Further, we structure P by defining $p +_1 q := (p \wedge q') + (p' \wedge q)$ for all $p,q \in P$. With this we can show

Theorem 2.12. A subset P of $[0,1]^S$ that fulfils (1) – (3) and for which the suprema and infima of any two elements exist is an algebra of S-probabilities if and only if $p' \wedge (q +_1 r) = (p' \wedge q) +_1 (p' \wedge r)$ for all orthogonal triples (p,q,r) of $(P, \leq, ')$, and $(P, \leq, ')$ is a Boolean algebra if and only if this is true for all $p,q,r \in P$.

Proof. If $p' \wedge (q +_1 r) = (p' \wedge q) +_1 (p' \wedge r)$ for all orthogonal triples (p,q,r) of $(P, \leq, ')$ then for such triples $p' \wedge (q +_1 r) = q +_1 r$ and hence, due to $q \perp r$, we have $q + r = q +_1 r \leq p'$ which implies (4).
Conversely, assume P to be an algebra of S-probabilities and let (p,q,r) be an orthogonal triple of $(P, \leq, ')$. Then $(P, \leq, ')$ is an orthomodular lattice and according to a well-known fact in lattice theory (cf. e. g. [4], Chapter II, Theorem 4.5) it follows that the subalgebra $(B, \leq, ')$ of $(P, \leq, ')$ generated by $\{p,q,r\}$ is a Boolean algebra. Taking into account that $p+q = p \vee q$ for $p,q \in P$ with $p \perp q$ we obtain for arbitrary elements f,g,h of B:

$$f \wedge (g +_1 h) = f \wedge ((g \wedge h') + (g' \wedge h)) = f \wedge ((g \wedge h') \vee (g' \wedge h)) = (f \wedge g \wedge h') \vee (f \wedge g' \wedge h)$$

and because of $(p \wedge q)' = p' \vee q'$, $((f \wedge g) \wedge (f \wedge h)') \perp ((f \wedge g)' \wedge (f \wedge h))$ and the distributive law which holds in Boolean algebras we have

$$(f \wedge g) +_1 (f \wedge h) = ((f \wedge g) \wedge (f' \vee h')) \vee ((f' \vee g') \wedge (f \wedge h)) = (f \wedge g \wedge h') \vee (g' \wedge f \wedge h).$$

This proves that $f \wedge (g +_1 h) = (f \wedge g) +_1 (f \wedge h)$ for all $f,g,h \in B$, in particular for $f = p'$, $g = q$ and $h = r$.
As for the second part of the proof, if P is a Boolean algebra then the above computation shows that $f \wedge (g +_1 h) = (f \wedge g) +_1 (f \wedge h)$ for all $f,g,h \in P$. Since $P = \{p' \mid p \in P\}$, this means that the equation $p' \wedge (q +_1 r) = (p' \wedge q) +_1 (p' \wedge r)$ holds for arbitrary $p,q,r \in P$.
Conversely, if $f \wedge (g +_1 h) = (f \wedge g) +_1 (f \wedge h)$ for all $f,g,h \in P$ then, in particular, $f \wedge (g +_1 g') = (f \wedge g) +_1 (f \wedge g')$ which implies that $f = (f \wedge g) \vee (f \wedge g')$ wherefrom we can conclude that all pairs of elements "commute" which means that $(P, \leq, ')$ is a Boolean algebra (cf. [4], see above). $\square$

3 Separating classical and quantum behaviour

The methods we present in the following are qualified for arbitrary systems of numerical events, however we point out some special features by studying the following cases:

Finite set of states If S is finite, say $S = \{s_1, \ldots, s_n\}$ with pairwise distinct $s_1, \ldots, s_n$ then $\{0,1\}^S$ can be interpreted as a set of n-tuples of 0 and 1 and the conditions (i) – (iii) of Theorem 2.1 for $\mathscr{M}$ take the following form for $\mathbf{P} = \{(p(s_1), \ldots, p(s_n)) \,|\, p \in P\}$:

(a) $(0, \ldots, 0) \in \mathbf{P}$

(b) If $(a_1, \ldots, a_n) \in \mathbf{P}$ then $(a_1, \ldots, a_n)' := (1-a_1, \ldots, 1-a_n) \in \mathbf{P}$.

(c) If $(a_1, \ldots, a_n), (b_1, \ldots, b_n) \in \mathbf{P}$ and $(a_1, \ldots, a_n) \wedge (b_1, \ldots, b_n) = (0, \ldots, 0)$ then $(a_1, \ldots, a_n) \vee (b_1, \ldots, b_n) \in \mathbf{P}$.

(Here $\vee$ and $\wedge$ are comprehended componentwise.)

If we replace (c) by

(c') For all $(a_1, \ldots, a_n), (b_1, \ldots, b_n) \in \mathbf{P}$ it follows that $(a_1, \ldots, a_n) \vee (b_1, \ldots, b_n) \in \mathbf{P}$.

then (a),(b) and (c') together imply that $(P, \leq, ')$ is a Boolean algebra.

If n is not too large it is easy to check, whether the above conditions are fulfilled. In particular, having already verified that P is an algebra of S-probabilities (conditions (a) – (c)) there is evidence of a classical physical system if the (componentwise) join of any two n-tuples of $\mathbf{P}$ belongs to $\mathbf{P}$.

Adding and subtracting of functions We assume a system of numerical events has been established which gives rise to an algebra of S-probabilities. Then one can detect a non-classical behaviour of a physical system by the following procedure due to Theorem 2.11:

For $f, g \in P$ check whether there exist $p, q \in P$ with $f - p \in P$, $p \perp q$ and $f + q = p + g$. If one finds a pair of numerical events $(f(s); s \in S_1)$ and $(g(s); s \in S_1)$ where S_1 is a subset of S such that no functions p and q exist with $f - p \in P$, $p \perp q$ and $(f(s) + q(s); s \in S_1) = (p(s) + g(s); s \in S_1)$ then it follows that one does not deal with a classical system of events.

Infinitely many states In this case we orientate ourselves on the Hilbert space model for algebras of S-probabilities mentioned in Example 1.3. In order to generalize this model we have assumed that the algebra P of S-probabilities is a lattice, which will be the assumption further on. Moreover, we endow P with the operation $+_1$ as we have done in Theorem 2.12.
Because of $(p \wedge q') \perp (p' \wedge q)$ it follows that $p +_1 q = (p \wedge q') + (p' \wedge q) = (p \wedge q') \vee (p' \wedge q)$. Therefore, in case of Boolean algebras $(P, \leq, ')$, $+_1$ is the symmetric difference $\triangle$ and can be interpreted from the logical point of view as "either – or". The operator $\wedge$ then corresponds to the logical "and".

Now let us assume that three measurements within a physical system result in probabilities $f(s), g(s), h(s)$ for a subset S_1 of S, $f, g, h \in P$. We denote the corresponding numerical events by F, G and H, i. e. $F = (f(s); s \in S_1)$, $G = (g(s); s \in S_1)$ and $H = (h(s); s \in S_1)$. If one observes that F occurs simultanously when either G or H occurs and the outcome is different when either F and G or F and H occur, then one knows that one does not deal with a classical physical system. The reason is that then $f \wedge (g +_1 h) \neq (f \wedge g) +_1 (f \wedge h)$ which must not be the case for a classical system according to Theorem 2.12. An example for such a situation would be the well-known experiment with filtering out a projection of "spin up" or "spin down" to the x- and y-axis of a rectangular coordinate system of electrons in a magnetic field (cf. [5]): Let us denote the occurrence of the event "spin up" of the x-component by A and the event "spin down" of the y-component by B. The experiment described in [5] is prepared in such a way that the x-component of "spin down" and the y-component of "spin up" are filtered out of the electron beam, so that the probability is 1 that $(B \wedge A) \triangle (B \wedge A')$ is true (since B and A are true), what could be verified by only detecting "spin up" of the x-component on a detecting screen. However, it turns out that also "spin down" of the x-component can be observed, so that $B \wedge (A \triangle A')$ is false and hence $(B \wedge A) \triangle (B \wedge A') \neq B \wedge (A \triangle A')$ contrary to their semantic equivalence within classical mechanics.

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